Letter to the Editor: Archimedes, Taylor, and Richardson

The enjoyable article "What if Archimedes Had Met Taylor?" (this MAGAZINE, October 2008, pp. 285–290) can be understood in terms of eliminating error terms. This leads to a different concluding approximation that is more in the spirit of the note by combining previous estimates for improvement. We denote the paper's weighted-average estimates for π using an n-gon by A_n using area and P_n using perimeter. The last section shows two formulas,

$$\text{Error(perim)} = P_n - \pi = \frac{\pi^5}{20n^4} + \frac{\pi^7}{56n^6} + \cdots \quad \text{and}$$
$$\text{Error(area)} = A_n - \pi = \frac{2\pi^5}{15n^4} + \frac{2\pi^7}{63n^6} + \cdots,$$

where we have corrected the first term in the latter. A combination of $8/5$ of the first and $-3/5$ of the second will leave $O(1/n^6)$ error. So, the last table could show

$$\frac{8}{5}P_{96} - \frac{3}{5}A_{96} = 3.14159265363.$$

This approach could be used to alternatively justify

$$A_n = \frac{1}{3}AI_n + \frac{2}{3}AC_n,$$

where AI_n and AC_n are inscribed and circumscribed areas respectively, by subtracting out the $1/n^2$ error terms and leaving the corrected error formula above. For general integration, a similar derivation motivates Simpson's rule as the combination of $1/3$ trapezoidal rule plus $2/3$ midpoint rule. This is more than a coincidence since the inscribed area connects arc endpoints as in trapezoidal rule and circumscribed area uses the arc midpoint.

The technique of combining estimates to eliminate error terms is known as Richardson's Extrapolation in most numerical analysis textbooks. It is usually applied to halving step-size in the same approximation formula. For example, Archimedes could have computed

$$\frac{16}{15}P_{96} - \frac{1}{15}P_{48} = 3.14159265337,$$

if Taylor could have whispered these magical combinations.

—–Richard D. Neidinger
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