Infinite Series Supplement

By Richard D. Neidinger
and Donna K. Molinek
Department of Mathematics
Davidson College
Davidson, NC 28036

Daily Outline:

Table of Contents

Application to Differential Equations from some DE textbook, for example:

Preface to the Instructor:

This unit is designed to supplement a differential equations course with an introduction to infinite series. It could be inserted between the method of undetermined coefficients (for linear DEs) and the chapter on series solutions of differential equations. Lessons emphasize understanding and manipulation of power series (or Taylor series). Throughout, geometric series are used as first examples where concepts are particularly simple, including convergence, partial sums, errors, radius and interval of convergence. Radius of convergence is hinted at in IS 1, explored graphically in IS 7, with ratio test in IS 9, and finally by distance to the nearest singular point of a differential equation in IS 12. This edition uses a standard ratio test, similar to what is found in Calculus texts.

The discussion of series convergence or divergence emphasizes the general definitions and important special cases, specifically geometric, harmonic, and alternating. Some topics are included primarily for the appreciation of series in contexts other that differential equations (particularly IS 3 and IS 6). Errors and error bounds, as well as partial sums, form a theme. The notation P_n is consistently used for the partial sum up to the term where n is the index value (as opposed to the first n nonzero terms), so it actually depends on the indexed expression for the series and the number of terms depends on the starting index. This has the advantage that if P_n is the nth degree Taylor polynomial, then P_n is also this nth partial sum. Any notation is inevitably confusing for a series with terms of value zero, but our choice eases the application to power series.

Sections are approximately one class, though IS 1 may spill into the second day since IS 2 is relatively short.

References: (in particular, sources of many homework exercises)

[ABD] H. Anton, I. Bivens, S. Davis, *Calculus, Early Transcendentals*, 8th edition, John Wiley, New Jersey, 2005.

[HG] D. Hughes-Hallet, A.M. Gleason, et. al., *Calculus*, John Wiley, New York, 1994.

[EP] C.H. Edwards Jr. and D.E. Penney, *Differential Equations and Boundary Value Problems, Computing and Modeling*, 4th edition. Prentice Hall, NJ, 2008.

■ Lesson IS 1: Taylor Polynomials

Solving second (and higher) order linear differential equations becomes the art of good guessing. For constant coefficients as in $y'' + 5y' - 2y = 0$, we guess $y = e^{rx}$ and derive the characteristic equation. For $y'' + 2y' + 3y = x^2$, we guess $y = Ax^2 + Bx + C$ in the method of undetermined coefficients. But variable coefficients make this much more difficult. What could we guess for $y'' + xy' + y = \ln(x-1)$? Even $y'' + xy + y = 0$ is a problem. The solution is to make the ultimate undetermined coefficients guess:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots.$$

It's amazing that practically any function $f(x)$ can be represented in this way! In this section, we will stop the infinite sum and show that $f(x)$ can be approximated by a polynomial

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

at least for x values near 0. In later sections, we will define what it means to add infinitely many terms and show that

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots,$$

so that the function actually equals the infinite series over some domain interval.

□ **Taylor Polynomials.** So what would it mean to have a polynomial approximate $f(x)$ near 0? If the polynomial stops at a *linear approximation* $f(x) \approx a_0 + a_1 x$, we would want the line to go through the point with the same slope as f. If it continues to a *quadratic approximation* $f(x) \approx a_0 + a_1 x + a_2 x^2$, then we would want the parabola to have the same concavity as f, so that the second derivatives should agree at 0. In general, we would like to have as many derivatives as possible agree at 0, and each new derivative determines another coefficient as we see in this table.

To make this approximation near 0		Require	
$f(x) \approx$	$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$	$f(0) =$	a_0
$f'(x) \approx$	$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$	$f'(0) =$	a_1
$f''(x) \approx$	$2a_2 + 3(2)a_3 x + 4(3)a_4 x^2$	$f''(0) =$	$2a_2$
$f'''(x) \approx$	$3(2)a_3 + 4(3)(2)a_4 x$	$f'''(0) =$	$3(2)a_3$
$f^{(4)}(x) \approx$	$4(3)(2)a_4$	$f'''(0) =$	$4(3)(2)a_4$

This results in the polynomial approximation formula

$$f(x) \approx f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \frac{1}{3!} f'''(0)x^3 + \frac{1}{4!} f^{(4)}(0)x^4$$

which clearly generalizes to an arbitrary number of terms. To get the n^{th} degree term, we would require that $f^{(n)}(0) = n! a_n$ or $a_n = \frac{1}{n!} f^{(n)}(0)$.

Of course, we may want the approximation near some other point a, instead of 0. The above method works well if we write the polynomial in powers of $(x-a)$. To make

$$f(x) \approx a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n \text{ near } a,$$

we require $f^{(n)}(a) = n! a_n$. Thus we should define $a_n = \frac{1}{n!} f^{(n)}(a)$. This leads to the general definitions.

Definition 1 *If f can be differentiated n times at $x = a$, then* <u>*the nth Taylor Polynomial*</u> *about a for f is*

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

This polynomial has the property that it best approximates f near $x = a$ in the sense that the values of its first n derivatives matches those of $f's$ at $x = a$.

Definition 2 *If $a = 0$, the resulting polynomial,*

$$p_n(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n$$

is called the nth Maclaurin polynomial for f.

Although the coefficients in these formulas look like large expressions, they always evaluate to just numbers as in the following examples.

Example 3 *Consider the function $f(x) = x \sin x$ near the point where $x = \pi$.*

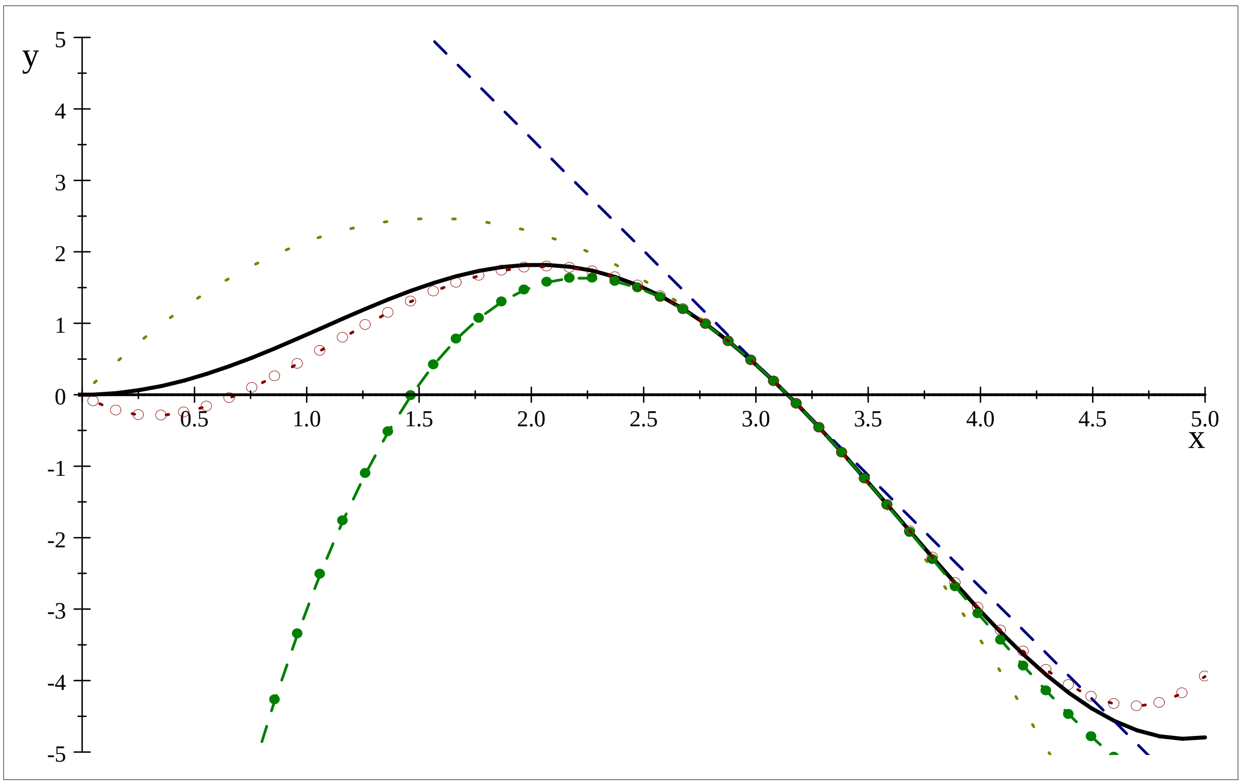
In the table below we compute several derivatives of $f$and corresponding coefficients.

derivatives of f	evaluated at $a = \pi$	$a_n = f^{(n)}(a)/n!$
$f(x) = x \sin x$	$f(\pi) = 0$	$a_0 = 0$
$f'(x) = \sin x + x \cos x$	$f'(\pi) = -\pi$	$a_1 = -\pi$
$f''(x) = 2\cos x - x \sin x$	$f''(\pi) = -2$	$a_2 = -2/2! = 1$
$f'''(x) = -3\sin x - x\cos x$	$f'''(\pi) = \pi$	$a_3 = \pi/3! = \pi/6$
$f''''(x) = -4\cos x + x\sin x$	$f''''(\pi) = 4$	$a_4 = 4/4! = 1/6$

So the 4th Taylor polynomial is

$$p_4(x) = -\pi\,(x-\pi) - (x-\pi)^2 + \frac{\pi}{6}\,(x-\pi)^3 + \frac{1}{6}\,(x-\pi)^4\,.$$

We could stop earlier and get simpler polynomials, including the linear approximation $p_1(x) = -\pi\,(x-\pi)$, quadratic approximation $p_2(x) = -\pi\,(x-\pi) - (x-\pi)^2$, and a cubic approximation $p_3(x) = -\pi\,(x-\pi) - (x-\pi)^2 + \frac{\pi}{6}\,(x-\pi)^3$. Here is a plot of these four polynomials together with the function.



Note in the graph, that as n increases, $p_n(x)$ gives a better approximation for $f(x)$ near $x = \pi$. This is because the n^{th} degree Taylor polynomial is calculated so that it will have the same value of f and its first n derivatives when they are evaluated at $x = \pi$.

Example 4 *Find the 6th Maclaurin polynomial for $f(x) = \sin x$.*

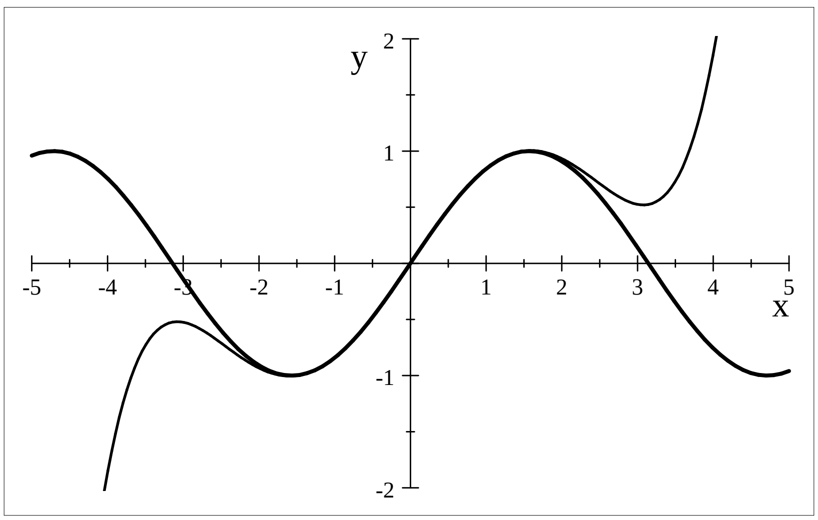
We first need to find $f, f', f'', \ldots, f^{(6)}$ and then evaluate each of these at $x = 0$ in order to find the coefficients $a_0, a_1, \ldots, a_6$ of the polynomial $p_n(x)$.

$f(x) = \sin x$	$f(0) = 0$	$a_0 = 0$
$f'(x) = \cos x$	$f'(0) = 1$	$a_1 = 1$
$f''(x) = -\sin x$	$f''(0) = 0$	$a_2 = 0$
$f^{(3)}(x) = -\cos x$	$f^{(3)}(0) = -1$	$a_3 = \frac{-1}{6}$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = 0$	$a_4 = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = 1$	$a_5 = \frac{1}{120}$
$f^{(6)}(x) = -\sin x$	$f^{(6)}(0) = 0$	$a_6 = 0$

So, the 6th Maclaurin polynomial for $f(x) = \sin x$ is

$$\begin{aligned} p_6(x) &= 0 + 1 \cdot x + 0 \cdot x^2 - \frac{1}{6}x^3 + 0 \cdot x^4 + \frac{1}{120}x^5 + 0 \cdot x^6 \\ &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \end{aligned}$$

Here is the graph of $f(x) = \sin x$ (bold) and $p_6(x)$:



It is appears that for $-2 \leq x \leq 2$, $p_6(x)$ is an extremely good approximation for $\sin x$. What is meant by this, is that if we want to approximate $\sin x$ for an x value near 0, we can simply compute $p_6(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$. For example $\sin(.3) = .295\,520\,21$ and $p_6(.3) = .295\,520\,25$. As another example, note that $\sin(1) = .841\,470\,98$ and $p_6(1) = .841\,666\,67$. Both are pretty good approximations, however, as x gets further from 0, the accuracy decreases.

The next theorem gives a formula for the error in estimating $f(x)$ by $p_n(x)$.

Theorem 5 (Taylor's Theorem) *If $f(x)$ has continuous derivatives through order $n+1$ on an open interval containing a and x and $p_n(x)$ is defined as above, then*

$$f(x) = p_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \textit{for some } c \textit{ between } x \textit{ and } a.$$

The number c is a theoretical point that we rarely know. Instead, we use it by bounding the "worst case" of how large the result could be. Define the approximation error $E_n(x) = f(x) - p_n(x)$. then

$$|E_n(x)| = \frac{\left|f^{(n+1)}(c)\right|}{(n+1)!}|x-a|^{n+1} \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

where M is greater than or equal to the maximum of $\left|f^{(n+1)}(x)\right|$ on an interval containing x and a.

Example 6 *Use Taylor's Theorem to bound the error in approximating* $\sin x$ *by* $p_6(x)$ *about* 0 *for* $x = .3$ *and for* $x = 1$.

For $f(x) = \sin x$, we found $p_6(x)$ and the approximations $p_6(.3)$ and $p_6(1)$ above. Using Taylor's theorem,

$$|\sin x - p_6(x)| \quad = \quad |E_6(x)| = \frac{\left|f^{(7)}(c)\right|}{7!} |x|^7 \leq \frac{M}{7!} |x|^7$$

To estimate M, observe that $f^{(7)}(x) = -\cos x$, so that $\left|f^{(7)}(c)\right| \leq M = 1$ on any interval whatsoever. For $x = .3$,

$$|\sin .3 - p_6(.3)| = |E_6(.3)| \leq \frac{1}{7!} |.3|^7 \leq 4.4 \times 10^{-8}$$

For $x = 1$,

$$|\sin 1 - p_6(1)| = |E_6(1)| \leq \frac{1}{7!} |1|^7 \leq .0002.$$

These error bounds are very close to the actual observed error above.

■ Exercises for Lesson IS 1

1. Find $p_3(x)$ about zero for $f(x) = e^x$ and find a general expression for $p_n(x)$.

2. Find $p_3(x)$ about zero for $f(x) = \arctan(x)$. (This is more typical of the difficulty in differentiating many times.)

3. Let $f(x) = \dfrac{1}{x}$.

   (a) Find the first 4 (p_0 to p_3) Taylor polynomials about $a = 1$ for $f(x)$.

   (b) Graph the polynomials and $f(x)$ on the same set of axes. If you have access to a program that automatically computes and plots higher-order Taylor polynomials, report the interval on which high-order polynomials appear to "fit."

   (c) Use $p_3(x)$ from part (a) to approximate $\frac{1}{1.4}$, $\frac{1}{1.02}$, and $\frac{1}{0.9}$. Compare the actual values.

   (d) All of the above x values should be in the interval $[.9, 1.4]$ and the maximum of $\left|f^{(4)}(x)\right|$ on this interval is less than or equal to $M = 41$ (can you verify this?). Using this M, bound the errors, $E_3(x)$ for each of the approximations in (c).

4. Let $g(x) = \cos x$.

   (a) Find the first 4 Taylor polynomials (p_0 to p_3) about $a = \pi/4$ for $g(x)$.

   (b) Use convenient upper estimates to bound the error $|E_3(x)|$ for any x in the interval $[.5, 1]$.

5. Let $f(x) = 7x^2 - 3x + 5$.

   (a) Find the first 4 Taylor polynomials (p_0 to p_3) about $a = 0$ for $f(x)$. Also, what is $p_{10}(x)$?

   (b) Find the 2^{nd} degree Taylor polynomial $p_2(x)$ about $a = 1$, so it will be in powers of $(x - 1)$.

   (c) Expand $p_2(x)$ out into powers of x. What do you notice?

   (d) The following statement is almost true; add one qualifying phrase to make it true: If $f(x)$ is any polynomial, then any corresponding Taylor polynomial $p_n(x)$ is equal to $f(x)$.

6. Construct the Maclaurin polynomial of degree 6 for $\cos x$. Based on the pattern, what are $p_7(x)$ and $p_8(x)$?

7. Suppose $p_2(x) = a + bx + cx^2$ is the second degree Taylor polynomial for the function $f(x)$ about zero. What can you say about the signs of a, b, and c if f has the graph given below?

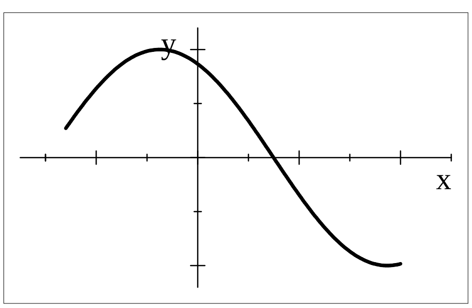


Figure for Exercise 7

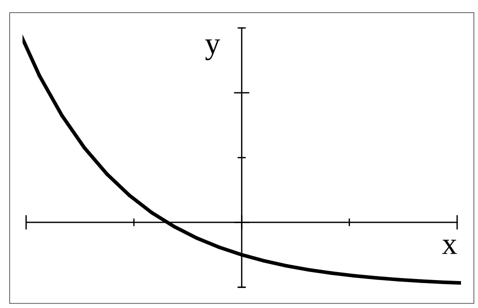


Figure for Exercise 8

8. Suppose $p_2(x) = a + bx + cx^2$ is the second degree Taylor polynomial for the function $f(x)$ about zero. What can you say about the signs of a, b, and c if f has the graph given above?

9. For $f(x) = \sin(x)$, the Maclaurin polynomial $p_1(x) = x$, so the error $E_1(x) = \sin(x) - x$.

   (a) Graph $\sin(x)$ and x together on $[-1, 1]$. How large does $|E_1|$ get on $[-.5, .5]$?

   (b) Use Taylor's theorem to bound $|E_1(x)|$ for any x in $[-.5, .5]$.

   (c) Why is it also true that $E_2(x) = \sin(x) - x$?

   (d) Use Taylor's theorem to bound $|E_2(x)|$ for any x in $[-.5, .5]$.

10. Euler's method, which is used to numerically solve a first order differential equation, can be thought of as using the first degree Taylor polynomial to approximate the unknown function $y(x)$. Recall that one step of Euler's method approximates $y(x_{n+1})$ by computing $y_{n+1} = y(x_n) + hy'(x_n)$. If we let $a = x_n$ and $x = x_{n+1} = a + h$, then Euler's method approximates $y(x) \approx y(a) + y'(a)(x - a)$. This is a Taylor polynomial approximation $y(x) \approx p_1(x)$ with error $|y(x) - p_1(x)| = |E_1(x)|$.

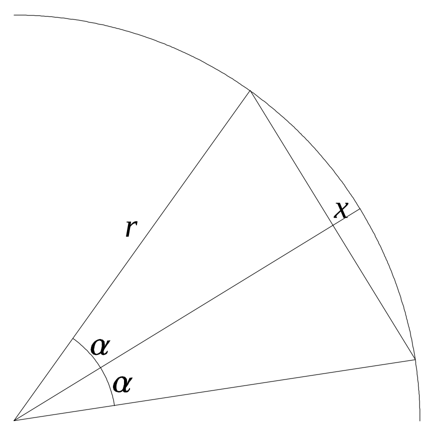
    (a) If we assume that $|y''(x)| \leq M$ for all x in our solution interval, what is a bound on the error from one step of Euler's method with $h = .1$? with $h = .01$? (in terms of M)

    (b) Typical Runge-Kutta methods work by approximating the function by $p_4(x)$. If we assume $\left|y^{(5)}(x)\right| \leq N$ for all x in our solution interval, what is a bound on the error from one step of the Runge-Kutta method with $h = .1$? with $h = .01$? (in terms of N)

    Note: This is a good example of how Taylor's Theorem is really used to compare methods.

11. For the function $f(x) = 1 - \cos(x)$, find the quadratic approximation $p_2(x)$ and compute the approximation for $x = 10^{-8}$. (This is an example where the approximation is actually more accurate than directly evaluating $1 - \cos(10^{-8})$ on most calculators, since they do not compute enough digits to distinguish between 1 and $\cos(10^{-8})$. The approximation avoids this pitfall, as does the equivalent expression $f(x) = \sin^2(x)/(1 + \cos(x))$.)

12. Consider a small slice of pie cut from a circle of radius r by angle 2α, as in this graphic.



    You will estimate x, the amount by which the arc extends out over the chord.

    (a) Write a formula for x using cos.

    (b) Use the quadratic approximation for cos about 0 to show that $x \approx r\alpha^2/2$.

    (c) Assuming the Earth is a sphere of radius 4000 mi, use the result in part (b) to approximate the amount by which a 100 mi arc along the equator will diverge from its chord.

13. Suppose that a function $f(x)$ has the same derivative values through order 5 as the polynomial $q(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + c_5(x - a)^5$ when you evaluate these derivatives at $x = a$. Equate the first derivatives at a, second derivatives at a, etc., and show how this gives formulas for c_0, c_1, c_2, c_3, c_4, and c_5 in terms of derivative values of f.

14. Find the nth Maclaurin polynomial for $f(x) = \frac{1}{1-x}$.

■ Lesson IS 2: Geometric Series.

Is it really possible for an infinite number of nonzero values to add up to a finite number? Certainly, we can add up a large finite number of terms. Fix some real number x and consider the special case of $P_{100} = 1 + x + x^2 + \cdots + x^{100}$. It is finite but is there any way to find the total sum without adding all 101 terms? Yes, simply multiply by x and subtract:

$$\begin{aligned} P_n &= 1 + x + x^2 + \cdots + x^n \\ xP_n &= x + x^2 + x^3 + \cdots + x^{n+1} \\ \text{so, } (1-x)P_n &= 1 - x^{n+1}. \\ \text{Thus, } P_n &= \frac{1 - x^{n+1}}{1-x}. \end{aligned}$$

What would it mean to add an infinite number of terms $1 + x + x^2 + \cdots + x^n + \cdots$? This is an infinite series also denoted $\sum_{n=0}^{\infty} x^n$. The infinite sum would have to be the limit of P_n as $n \to \infty$. This, depends on what happens to x^{n+1} as as $n \to \infty$. If $|x| > 1$, then x^n grows without bound as $n \to \infty$, so the infinite sum really doesn't sum to a finite number. We say the series *diverges* for $|x| > 1$. However, if $|x| < 1$, then $x^n \to 0$ as $n \to \infty$.

$$\begin{aligned} \sum_{n=0}^{\infty} x^n &= \lim_{n\to\infty} P_n \\ &= \lim_{n\to\infty} \frac{1 - x^{n+1}}{1-x} \\ &= \frac{1-0}{1-x} \quad \text{for} \quad |x| < 1. \end{aligned}$$

This formula doesn't make sense if $x = 1$, but the infinite series $1 + 1 + 1 + \cdots$ clearly diverges. Also, if $x = -1$, then the limit above does not exist (it oscillates). In conclusion, the Geometric Series

$$\begin{aligned} \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad &\text{for} \quad |x| < 1 \qquad , \\ \text{but diverges for} \quad &|x| \geq 1 \qquad . \end{aligned}$$

Sometimes it takes some thought to decide if a series is geometric or not. Consider the series

$$\frac{7y^2}{2} - \frac{7y^3}{4} + \frac{7y^4}{8} - \frac{7y^5}{16} + \cdots.$$

It is *geometric* if each term is the same multiple of the previous term. In this case, multiplying each term by $-y/2$ gives the next term, we say the ratio between terms is $x = -y/2$. Then, factor out the first term to see the geometric series.

$$\begin{aligned} & \frac{7y^2}{2} - \frac{7y^3}{4} + \frac{7y^4}{8} - \frac{7y^5}{16} + \cdots \\ = & \frac{7y^2}{2}\left(1 + \left(\frac{-y}{2}\right) + \left(\frac{-y}{2}\right)^2 + \left(\frac{-y}{2}\right)^3 + \cdots\right) \\ = & \frac{7y^2}{2}\sum_{n=0}^{\infty}\left(\frac{-y}{2}\right)^n \\ = & \frac{7y^2}{2}\frac{1}{1-(-y/2)} \\ = & \frac{7y^2}{2+y}. \end{aligned}$$

Finite geometric sums are always of the form where each term is the same multiple of the previous term. In the above example with ratio $x = -y/2$ and $a = 7y^2/2$, the sum could stop after some number of terms and be represented as $P_n = a + ax + ax^2 + \cdots + ax^n$. This would sucumb to the "multiply by x and subtract" trick that began this seciton. The result has a form that may facilitate memory and application

$$\begin{aligned} P_n &= a + ax + ax^2 + \cdots + ax^n \\ &= \frac{a - ax^{n+1}}{1-x} \\ &= \frac{\text{first term} \ - \ \text{term after last}}{1-x} \quad \text{for } x \neq 1 \end{aligned}$$

In our formula for the infinite geometric sum, the starting point is $n = 0$. If the series starts at k, then the final sum is not given by $\dfrac{1}{1-x}$, instead we need to adjust in one of the following ways.

Example 7 *Find the sum of* $\displaystyle\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n$.

First this is a geometric series with $x = 2/3$, so it converges. To use the formula, we need to have the starting point be zero.

$$\begin{aligned} \sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n &= \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n - \sum_{n=0}^{2} \left(\frac{2}{3}\right)^n \\ &= \frac{1}{1-\frac{2}{3}} - \left(1 + \frac{2}{3} + \frac{4}{9}\right) \\ &= 3 - \frac{19}{9} \\ &= \frac{8}{9} \end{aligned}$$

An alternate way to find this infinite sum is to factor the first term out, in expanded form, as was done for the y example above.

$$\begin{aligned} \sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n &= \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^5 + \cdots \\ &= \left(\frac{2}{3}\right)^3 \left(1 + \left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 + \cdots\right) \\ &= \left(\frac{2}{3}\right)^3 \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \\ &= \left(\frac{2}{3}\right)^3 \left(\frac{1}{1-2/3}\right) \\ &= \frac{8}{9}. \end{aligned}$$

A more formal way to accomplish the above is by a technique we will find very useful later, called change of indices. We substitute $i + 3$ for every n in the series expression:

$$\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n = \sum_{i+3=3}^{\infty} \left(\frac{2}{3}\right)^{i+3} = \left(\frac{2}{3}\right)^3 \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^i = \left(\frac{2}{3}\right)^3 \left(\frac{1}{1-2/3}\right) = \frac{8}{9}.$$

We have begun to do some algebra on series. The following identities are valid whenever each series converges.

$$\begin{aligned}\sum_{n=k}^{\infty} ac_n &= a\sum_{n=k}^{\infty} c_n \quad \text{and}\\ \sum_{n=k}^{\infty} (a_n + b_n) &= \sum_{n=k}^{\infty} a_n + \sum_{n=k}^{\infty} b_n.\end{aligned}$$

While these may seem obvious, the second is a kind of infinite commutative property which can lead to false results if any of the parts actually diverge. Convergence and divergence are discussed in IS4.

□ IS 2 Homework

1. Find the sum of $\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\cdots+\frac{3}{2^{10}}$
2. Suppose you take the October calendar and put 1¢ on the first 2¢ on the second, 4¢ on the third and continue to double the money on each consecutive day in the month. If you could actually accomplish this, how much money would be sitting on the calendar after you finished?

In exercises 3 - 8, write each in summation notation and tell if it is a geometric series. For those which are, find a (closed form, i.e. an ordinary formula, not using $\sum$ or $+\cdots$.) formula for the infinite sum.

3. $3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\cdots$
4. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$
5. $e^{-2}+e^{-3}+e^{-4}+e^{-5}+\cdots$
6. $1-y^2+y^4-y^6+\cdots$
7. $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots$
8. $\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}+\cdots$

In exercises 9 and 10, find the indicated sum.

9. $\displaystyle\sum_{n=4}^{\infty}\frac{1}{3^n}$
10. $\displaystyle\sum_{n=0}^{\infty}\frac{3^n+5}{4^n}$
11. A repeating decimal can always be expressed as a fraction. Use the fact that

$$0.272727\ldots \quad = \quad 0.27+0.0027+0.000027+\cdots$$

to write this number as a geometric series. Find the sum as an exact fraction.

12. Prove $0.9999\ldots=1$.
13. A ball shot up from the ground rises to a height of 10 feet, falls and bounces. Each bounce is 60% of the height of the bounce before. Let D_n = the total vertical distance (up plus down) the ball has traveled when it hits the floor for the nth time.
    (a) Find an expression for D_n and find a closed form for the value. (Check: $D_5 \approx 46.1$ ft.)
    (b) What is the total vertical distance traveled after infinitely many bounces?
    You may think that the ball keeps bouncing forever since it takes infinitely many bounces. This is not true! To help show this, we do a physical DE problem in (c).
    (c) Show that a ball dropped from a height of h feet reaches the ground in $\sqrt{h}/4$ seconds. Start with the DE for the position $x(t)$ of the ball: $x''=-32$ (ft/sec^2 due to gravity). Solve the initial value problem for dropping from height h. Then solve for the time it takes to hit the ground.
    (d) Assume that it takes the same amount of time to rise up to a height h as to fall from that height. For the original ball shot up from the ground, write a geometric series for the total time this ball takes to complete the infinitely many bounces in (b). Find how many seconds until this ball stops bouncing.

■ Lesson IS 3: Time value of money

Time value of money is a valuable life lesson, though only relevant to our course as an application of finite geometric series. Everyone needs to understand that amounts of money at different times in the future cannot be compared by simply adding up the amounts. A contract for \$100 two years from now is not "worth" the same amount as \$100 today. Think of money as being placed at different points along a time line. To compare the amounts you must "pull" each amount to the same time, by multiplying by a factor of $(1+i)^n$ to bring it forward n periods or $(1+i)^{-n}$ to bring it backward n periods. Here, i is an interest rate that can be thought of as account earnings, or debt interest rate, or even inflation rate. They all mean that \$100 today would be traded for more than that in the future; and \$100 in the future is worth less than that now.

Consider the scenario where you have a choice between \$1,000 today or two amounts, \$500 one year from now plus \$600 five years from now.

How much would the \$1,000 be worth five years from now? Suppose you could invest it in an account earning 5% per year. Every year, you keep what you had at the beginning plus .05 times that balance. Your account would accumulate to

after one year: $\$1,000 + .05 * \$1,000 = \$1,000(1 + .05) = \$1,050$,

after two years: $\$1,050 + .05 * \$1,050 = \$1,050(1 + .05) = \$1,000(1 + .05)^2 = \$1,102.50$,

etc., to after five years: $\$1,000(1 + .05)^5 = \$1,276.28$.

Now consider the two amounts. The \$600 that is given five years from now will be worth \$600 at that point in time. However, we can't simply add the \$500 one year from now. We bring the \$500 forward in time four years, so that all amounts are compared at this future point five years from now. Then the total of the two amounts is worth $\$500(1 + .05)^4 + \$600 = \$1,207.75$. This is not quite as good as the $\$1,276.28$ value of \$1,000 today, the better deal. Of course, this depends on the interest rate. If there was no interest, then the $\$500 + \$600$ would be a better deal. There must be a crossover point where the deals are equal. We can solve the equation numerically to find i when $\$1,000(1 + i)^5 = \$500(1 + i)^4 + \$600$. The solution $i = 3.1\%$ is the interest rate we would be charging if we gave a friend \$1,000 today in exchange for this two amount deal.

In general, the *future value* of $\$A$ after n compounding periods is

$$F = A(1+i)^n$$

where,

$$i = \text{interest rate per period, so}$$
$$i = \frac{r}{m} = \frac{\text{nominal annual interest rate}}{\text{number of periods per year}}.$$

For example, if you put the \$1,000 in a bank account earning 2% compounded daily, then the bank will will pay .02/365 times your balance every day. After five years, this would result in $\$1,000(1 + .02/365)^{5*365} = \$1,105.17$.

While money at different times must be brought to the same time for comparison, that time does not have to be in the future and is most commonly chosen to be the present. Indeed, we can compare how much "present value" one would need today to generate certain amounts in the future. At 5% annual interest, how much $\$P$ do you need today to result in \$600 in five years? The equation would be $\$600 = P(1 + .05)^5$ or $P = \$600(1 + .05)^{-5}$. All these earnings in investment accounts are hypothetical and do not actually have to be done, but they do give a consistent way to compare. In general, the *present value* of $\$A$ coming n compounding periods in the future is

$$P = A(1+i)^{-n}$$

where $i = \dfrac{r}{m}$, the interest rate per period as before.

For original scenario, let's compare the present values at 5% annual interest . Clearly the \$1,000 today has a present value of \$1,000. The present value of the two amount deal is $\$500(1+.05)^{-1} + \$600(1+.05)^{-5} = \$946.31$. Again, we conclude that the \$1,000 today is the better deal. The conclusion must be consistent with the above future value analysis; in fact, we could multiply these present values by the same constant $(1+.05)^5$ to get the future values above. Likewise, we could numerically solve for the interest rate to equate these in present value $\$500(1+i)^{-1}+\$600(1+i)^{-5}=\$1000$, an equivalent equation to before with the same solution $i=3.1\%$. Since the earnings are hypothetical and comparisons can be done at any time, present value is the most popular choice, so maybe "there is no time like the present."

□ Present Value of Regular Payments

Now, let's consider the present value of a series of regular payments, sometimes called an annuity. As an example, consider the Spring 2008 advertisement for a Chrysler Sebring Convertible for \$259/month. The fine print says "starting from" so we will be talking about he cheapest "stripped down" model that few would actually buy. The finer print says that this requires \$2988 cash down plus \$699 doc fee. If we take this deal, how much we are actually paying for the car? The answer is the present value of all payments at the 6.9% interest rate that the dealer charges, which equals the number of cash dollars that the dealer would take today in lieu of this deal.

Consider a time line with 36 payments of $A=\$259$ each, spaced one month apart, starting with the first payment one month from now. We need to sum the present value of each of those payments at monthly interest rate $i=0.069/12=.00575$, so we get

$$P=A(1+i)^{-1}+A(1+i)^{-2}+\cdots+A(1+i)^{-35}+A(1+i)^{-36}.$$

This is a finite geometric series, so we could use a formula, but it is almost as simple (and easier to remember and understand) to just use the multiply and subtract trick.

$$\begin{aligned}
P &= A(1+i)^{-1}+A(1+i)^{-2}+\cdots+A(1+i)^{-35}+A(1+i)^{-36}\\
(1+i)P &= A \quad +A(1+i)^{-1}+\cdots+A(1+i)^{-34}+A(1+i)^{-35}\\
(1+i)P-P &= A-A(1+i)^{-36}\\
iP &= A(1-(1+i)^{-36})
\end{aligned}$$

The result is

$$P=\$259\frac{\left(1-(1+.00575)^{-36}\right)}{.00575}=\$8,400.53.$$

Add this to the \$2988 down payment to get \$11,388.53 for the car, plus \$699 doc fee, plus tax, tag and title. This agrees with a calculator on the dealer's website that will calculate how much car you can afford for monthly payments and down payment that you specify.

If you can afford to pay cash for the car, should you opt for the monthly payments? No, not unless you can earn more than the 6.9% interest rate on your investments. Assuming you are earning 2% compounded monthly, the present value "to you" of the payments is

$$P=\$259\frac{\left(1-(1+.02/12)^{-36}\right)}{.02/12}=\$9,042.48.$$

This is the amount you would have to have in your 2% account now to exactly make all of the \$259 payments from just this savings account. Clearly, the payments cost you more than just paying the \$8,400.53 cash.

In general, the <u>*present value of an annuity*</u> is the present value of regular payments of $\$A$ at the end of each of n periods and is given by the formula

$$P=A\frac{(1-(1+i)^{-n})}{i}.$$

Anyone regularly using this concept knows this formula and probably has it as a calculator button. For those with infrequent need for it, the ideas of present value, together with the multiply and subtract trick, are powerful tools for remembering and understanding such payments. One can also use these tools to adapt to payments that don't exactly fit the standard setup.

□ Future Value of Regular Payments

Let's use these tools to derive the future value of regular payments, as you might to in a savings plan. Suppose you save $\$A$ per period with the first deposit being one period from now and continuing at the end of each of n periods. How much will you accumulate at the end of the n periods? We sum the future value of each of the payments at the time of the last payment.

$$\begin{aligned}
F &= A(1+i)^{n-1} + A(1+i)^{n-2} + \cdots + A(1+i)^2 + A(1+i)^1 + A \\
(1+i)F &= A(1+i)^n + A(1+i)^{n-1} + \cdots + A(1+i)^3 + A(1+i)^2 + A(1+i) \\
(1+i)F - F &= A(1+i)^n - A \\
iF &= A((1+i)^n - 1)
\end{aligned}$$

Resulting in the *future value of an annuity*: the future value of regular payments of $\$A$ at the end of each of n periods at the time of the last payment is

$$F = A\frac{((1+i)^n - 1)}{i}.$$

Let's consider the cash or loan example from a future value perspective. Suppose you have \$8,400.53 in savings and will have disposable income of \$259/month for 3 years, so you really have the choice of paying cash or these payments for the car. Again, assume the savings account earns 2% compounded monthly. If you choose the loan, money stays in savings and payments go toward the car; so, at the end, you will have $\$8,400.53(1+.02/12)^{36} = \$8,919.54$ in the bank and a 3-year old car. If you choose to pay cash, your savings start at zero but you can deposit the \$259/month in your account; so, at the end, you will have $\$259((1+.02/12)^{36} - 1)/(.01/12) = \$9,601.16$ in the bank and a 3-year old car. Of course, the conclusion is the same, pay cash; in fact, these future values are just the present values of $\$8,400.53$ and $\$9,042.48$ brought forward by $(1+.02/12)^{36}$.

Actually, a savings plan might be better conceived as deposits at the beginning of each period. Exercise 6 asks you to find this is slightly different formula.

□ Exercises for Lesson IS 3.

1. Your \$100 today will be worth how much three years in the future at 4% interest? Compare the amounts for annual, monthly and daily compounding.

2. Someone who needs cash now promises to pay you \$10,000 that they will get on their 21st birthday which is 30 months from now. If you want to charge them 5% interest, how much money should you give them today? Assume monthly compounding.

3. How much do you have to have in the bank on March 1, earning 3% interest compounded monthly to be able to take out \$3000 on June 1, \$30,000 on Aug. 1, and \$30,000 on Jan. 1?

4. You win a lottery prize of \$3,000,000! (That's a surprise ! not a factorial!) What this really means is that you can get 100,000 per year for 30 years. Actually, that is really \$70,000 per year for 30 years after taxes. They offer you a lump sum instead of \$1,500,000 or \$1,050,000 after taxes. Assume that with this much money you could earn 8% compounded annually. What is the present value of the \$70,000 per year offer? Do you take the lump sum? Bonus: Solve numerically for the interest rate at which the \$70,000 per year would exactly equal the lump sum offer.

5. You want to fund a college fund for a relative that will begin college in 5 years. You want to put in just the right amount on July 1, 2009 so that the student can withdraw \$10,000 on each of July 1 and January 1 of four years starting July 1, 2014. Assuming the fund will earn 5% compounded semi-annually, how much do you put in?

6. Derive a formula for the future accumulation of deposits of $\$A$ done at the *beginning* of each of n periods. What will they be worth at the end of the n periods? Use rate i per period. Start with by finding the finite geometric series and derive the sum formula by using the "multiply and subtract trick."

7. You wish to finance a home loan of \$200,000 over 30 years at 6% interest (computed monthly). What will be your monthly payment?

8. Suppose you want to endow a scholarship that will grant \$10,000 per year forever! Surprisingly, this has a finite present value. Let i denote annual interest rate earned by the endowment. What is the present value of the first \$10,000 grant that will come one year from now? The second \$10,000 two years from now? Write an $(+\cdots)$ expression for the present value P of all infinitely many \$10,000 grants. Use the "multiply and subtract trick" to find a simple formula for P. The final answer should be surprisingly simple (until you think about it). For $i = 5\%$, what is P (the amount you would have to donate to make this really happen!)?

■ Lesson IS 4: General Convergence of Series

In this section, we define and discuss various lists (sequences) of numbers associated with an infinite series, $\sum a_n$:

- the sequence of terms a_n,
- the associated partial sums, P_n, and
- the error terms E_n, (if given a conjecture about the sum of the series);

and the corresponding limits of these sequences:

- $\lim a_n$,
- $\lim P_n$, and
- $\lim E_n$.

We first define these terms and then discover how each of these three limits are related to the infinite series $\sum a_n$. To this end, we give a few definitions:

Definition 8 *An (infinite) series is a summation of the form* $a_k + a_{k+1} + a_{k+2} + \cdots + a_j + \cdots = \sum_{n=k}^{\infty} a_n$. *Usually,* $k = 0$ *or* 1 *but the starting point can be any integer.*

Definition 9 *The* n^{th} *term of the series is* a_n.

Definition 10 *The limit of (the sequence of) the terms is* $\lim_{n\to\infty} a_n$.

Definition 11 *The partial sum up to n is* $P_n = \sum_{i=k}^{n} a_i$.

Definition 12 *The limit of the sequence of partial sums is* $\lim_{n\to\infty} P_n$.

We illustrate each of the above definitions with an example. Consider the infinite series

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{n^2} = \frac{-1}{9} + \frac{1}{16} - \frac{1}{25} + \cdots + \frac{(-1)^n}{n^2} + \cdots$$

The nth term of the series, $a_n = \frac{(-1)^n}{n^2}$. The sequence of terms is

$$\frac{-1}{9}, \frac{1}{16}, \frac{-1}{25}, \ldots, \frac{(-1)^n}{n^2}, \ldots.$$

The limit of the sequence of terms of the series is

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0.$$

The partial sum up to say 9 (i.e.,use $n = 9$) is

$$P_9 = \frac{-1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \frac{1}{49} + \frac{1}{64} - \frac{1}{81} = -\frac{495\,091}{6350\,400} = -0.077962.$$

The sequence of partial sums is:

$$P_3, P_4, P_5, P_6, P_7, P_8, P_9, \ldots, P_n, \ldots.$$

and the corresponding values (P_3 to P_9 only) are:

$$-.1111,\ -0.0486,\ -0.0886,\ -0.0608,\ -0.0812,\ -0.0656,\ -0.07796, \ldots.$$

It appears that $\lim_{n\to\infty} P_n = -0.0725$. (This can be verified by computing P_n for very large values of n.) As we will see, it is often very difficult (or impossible) to find a formula for P_n and we need to rely on numerical approximations to find this important limit.

☐ Instructions for numerical computation of partial sums with selected technological tools.

- To compute $P_n = \sum_{i=0}^{n} a_i$ on **Mathematica**:

  Clear[P]

  P[n_] := NSum[a_i, $\{i, 0, n\}$]   (Shift + Enter defines this function)

  P[25]   (returns P_{25})

- On TI-89: $\sum(a_i, i, 0, n)$ ($\sum$ is on the F3 Calc menu)

Values of sequences and series can always be approximately numerically. Often numerical estimation is the only practical way to find the limit of the partial sums, $\lim_{n\to\infty} P_n$. The quantity E_n, called the error term, is the difference between the conjectured sum of the series and the partial sums. More about these later, right now, use the example to get a feel for the terms discussed so far.

Example 13 (1) $\sum_{n=0}^{\infty} \frac{1}{n!}$. *Tabulate the values of* a_n, P_n, *and* E_n *for the first five* n *values,* $n = 25$, *and* $n = 100$ *and then estimate the three limits* $\lim a_n$, $\lim P_n$, *and* $\lim E_n$ *numerically.*

Note in the following table that the value of P_n for large values of n is 2.7183. We conjecture that the limit is actually the number e. We will use the value of e for computing E_n

Solution 14

	a_n	P_n	E_n
$n = 0$	1	$\sum_{n=0}^{0} \frac{1}{n!} = 1$	$e - \sum_{n=0}^{0} \frac{1}{n!} = 1.7183$
$n = 1$	1	$\sum_{n=0}^{1} \frac{1}{n!} = 2$	$e - \sum_{n=0}^{1} \frac{1}{n!} = .7183$
$n = 2$	$\frac{1}{2!} = .5$	$\sum_{n=0}^{2} \frac{1}{n!} = 2.5$	$e - \sum_{n=0}^{2} \frac{1}{n!} = .2183$
$n = 3$	$\frac{1}{3!} = .16667$	$\sum_{n=0}^{3} \frac{1}{n!} = 2.6667$	$e - \sum_{n=0}^{3} \frac{1}{n!} = 5.1633 \times 10^{-2}$
$n = 4$	$\frac{1}{4!} = .041667$	$\sum_{n=0}^{4} \frac{1}{n!} = 2.7083$	$e - \sum_{n=0}^{4} \frac{1}{n!} = 9.948495 \times 10^{-3}$
$n = 10$	$\frac{1}{10!} = 2.7557 \times 10^{-7}$	$\sum_{n=0}^{10} \frac{1}{n!} = 2.718281$	$e - \sum_{n=0}^{10} \frac{1}{n!} = 2.7 \times 10^{-8}$
$n = 15$	$\frac{1}{15!} = 7.6471637 \times 10^{-13}$	$\sum_{n=0}^{15} \frac{1}{n!} = 2.718281$	$e - \sum_{n=0}^{15} \frac{1}{n!} =$ "0"
$n = 25$	$\frac{1}{25!} = 6.4 \times 10^{-26}$	$\sum_{n=0}^{25} \frac{1}{n!} = 2.7183$	$e - \sum_{n=0}^{25} \frac{1}{n!} =$ "0"
$n = 100$	$\frac{1}{100!} = 1.1 \times 10^{-158}$	$\sum_{n=0}^{100} \frac{1}{n!} = 2.7183$	$e - \sum_{n=0}^{100} \frac{1}{n!} =$ "0"
limit as $n \to \infty$	0	2.7183 *or* e?	0

□ The relation between $\lim P_n$ and $\sum a_n$.

In this section, we give the definition of series convergence.

Definition 15 *The series $\sum a_n$ converges if the limit of partial sums converges to a real number.*

If the series converges (that is, $\lim_{n\to\infty} P_n = L$), then we write $\sum_{n=k}^{\infty} a_n = L$ and say the series converges to L or the infinite sum equals L. If $\lim_{n\to\infty} P_n$ diverges, we say the series $\sum_{n=k}^{\infty} a_n$ diverges.

Although the actual values of P_n and L depend on k, the convergence of the infinite series does not depend on k. Therefore, if the same series is started at two different integers and one of them converges, then the other must also converge. The infinite sums of the two series differ by the few terms that one has and the other does not. In other words, the convergence or divergence of an infinite series depends on the "tail" $\left(\sum_{n=M}^{\infty} a_n\right)$ of the series and not the starting point. For this reason, we sometimes write $\sum a_n$ converges (or diverges) when an infinite sum is assumed and the starting point is irrelevant.

□ Errors and the relation between $\lim E_n$ and $\sum a_n$ (Vanishing Error Theorem).

The error, E_n is the difference between the partial sums and the sum of the series. It gives how far the partial sums are from the sum of the series. This is made more precise in the following:

Definition 16 *If the series is conjectured to converge to L, we define the error term, $E_n = L - P_n$.*

Definition 17 *The limit of the sequence of error terms is $\lim_{n\to\infty} E_n$.*

We note that if the infinite series $\sum a_n$ converges, the sum L is given by $\lim_{n\to\infty} P_n$. That is, the partial sums are getting closer to the sum L and, hence, the difference between the two (given by E_n) should be getting very small. This motivates the following theorem that is useful in proving the convergence of Taylor series.

Theorem 18 (Vanishing Error) *An infinite series converges to L if and only if the limit of the error terms is zero, i.e.*

$$\sum_{n=k}^{\infty} a_n = L \quad \textit{iff} \quad \lim_{n\to\infty} E_n = 0 \quad \textit{iff} \quad \lim_{n\to\infty} |E_n| = 0$$

Proof. $\sum_{n=k}^{\infty} a_n = L$ iff $\lim_{n\to\infty} P_n = L$ iff L$-\lim_{n\to\infty} P_n = 0$ iff $\lim_{n\to\infty} (L - P_n) = 0$ iff $\lim_{n\to\infty} E_n = 0$ iff $\lim_{n\to\infty} |E_n - 0| = 0$ iff $\lim_{n\to\infty} |E_n| = 0$. ■

□ The relation between $\lim a_n$ and $\sum a_n$ (Divergence Test).

In order for the series to converge, intuitively it makes sense that the values of the terms (a_n) must be getting smaller. If the terms are not getting smaller, then as P_n is computed, larger numbers keep getting added on to the partial sums. It seems clear in this case, that the partial sum sequence cannot converge, but will get arbitrarily large. However, if the terms are getting smaller (even approaching 0), this does not mean the series will converge. Consider the following two examples:

Example 19 (2) *Tabulate the values of a_n and P_n for several values of n to estimate the limits $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} P_n$. Do this for*

a. $\sum_{n=1}^{\infty} \frac{2n}{2n-1}$ *and* **b.** $\sum_{n=1}^{\infty} \frac{2n}{2n^2-1}$.

		$\sum_{n=1}^{\infty} \frac{2n}{2n-1}$		$\sum_{n=1}^{\infty} \frac{2n}{2n^2-1}$
n	a_n	P_n	a_n	P_n
1	2	2	2	2
5	1.1111	6.7873	.20408	3.3865
10	1.0526	12.133	.1005	4.0381
50	1.0101	52.938	.020004	5.6106
100	1.005	103.28	.010001	6.2988
200	1.0025	203.63	.0050001	6.9895
500	1.001	504.09	.002	7.9043

The P_n for each series, seems to be getting larger. However, the first series seems most clearly to diverge to infinity in the infinite sum. Note that in the first series, $\lim_{n\to\infty} a_n = 1$ and in the second series, $\lim_{n\to\infty} a_n = 0$.

This example motivates the following theorem and the warning following it. Notice that the proof of this theorem uses a contrapositive argument.

Theorem 20 (Divergence Test) *If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges.*

Proof. The theorem is logically equivalent to its contrapositive: "If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$." To prove this, suppose that $\sum a_n$ converges, so $\sum_{n=k}^{\infty} a_n = L$. Then $\lim_{n\to\infty} P_n = L$.

Also, $\lim_{n\to\infty} P_{n-1} = L$. Thus, $\lim_{n\to\infty} P_n - P_{n-1} = L - L = 0$.

But, $P_n - P_{n-1} = a_n$. So, $\lim_{n\to\infty} a_n = 0$. ■

WARNING: If $\lim_{n\to\infty} a_n = 0$ then convergence of $\sum a_n$ is inconclusive!! The series may converge or diverge; another method must be used to decide.

In our example, we can say that (a) $\sum_{n=1}^{\infty} \frac{2n}{2n-1}$ diverges by the divergence test, since $\lim_{n\to\infty} \frac{2n}{2n-1} = 1 \neq 0$. However, the divergence test is inconclusive for (b) $\sum_{n=1}^{\infty} \frac{2n}{2n^2-1}$., since $\lim_{n\to\infty} \frac{2n}{2n^2-1} = 0$. The slowly growing infinite sum of (b) may or may not approach a limit, we need techniques in a later lesson to settle the issue.

□ **Principles for evaluating limits theoretically.**

With a formula for a_n (always available) or for P_n (rarely available), the theoretical limit can be found. The case where $\sum a_n$ is a geometric series is an important exception where the formula for P_n is always known. Thus, the convergence and sum (if the series converges) of $\sum a_n$ can always be determined. To find the exact (theoretical) limit of an expression, the following principles (theorems) are useful:

- $\lim\limits_{n\to\infty} \dfrac{1}{g_n} = 0$ whenever $\lim\limits_{n\to\infty} g_n = \infty$.
- $\lim\limits_{n\to\infty} x^n = 0$ if $|x| < 1$ but $\lim\limits_{n\to\infty} x^n \neq 0$ if $|x| \geq 1$
- $\lim\limits_{n\to\infty} \dfrac{\text{polynomial of } n}{\text{polynomial of } n} = \lim\limits_{n\to\infty} \dfrac{(\text{polynomial of } n)\left(\frac{1}{n^k}\right)}{(\text{polynomial of } n)\left(\frac{1}{n^k}\right)} = \dfrac{0, \text{leading coefficient, or } \infty}{\text{leading coefficient}}$ where n^k is the highest denominator power.
- $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$ for all x.
- $\lim\limits_{n\to\infty} \dfrac{n^p}{x^n} = 0$ for $x > 1$ and for all p.
- $\lim\limits_{n\to\infty} \dfrac{\ln n}{n^p} = 0$ for all $p > 0$.

The last three items are rigorous ways to express what many intuitively know - that factorial growth ($n!$) is faster than exponential growth (x^n). Similarly, exponential growth is faster than polynomial growth (n^p) which is faster than logarithmic growth ($\ln n$). Remember, the n is the quantity going to ∞.

Remark 21 *All of the above limit principles hold true with $n+1$ in place of n, i.e., $\lim\limits_{n\to\infty} g_{n+1} = \lim\limits_{n\to\infty} g_n$ since they represent exactly the same sequence of terms.*

Example 22 (3) *For $\sum\limits_{n=0}^{\infty} \frac{3}{5^n}$, find the following limits and each corresponding conclusion about the series.*

(i) $\lim\limits_{n\to\infty} a_n$

(ii) $\lim\limits_{n\to\infty} P_n$

(iii) $\lim\limits_{n\to\infty} E_n$

(i) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{3}{5^n} = 3 \lim\limits_{n\to\infty} \left(\frac{1}{5}\right)^n = 0$.

The convergence of the series $\sum\limits_{n=0}^{\infty} \frac{3}{5^n}$ is inconclusive from the warning following the divergence test. (That is, the divergence test gives us no information and we need to do more to determine whether the series converges or diverges.)

(ii) Note that $P_n = 3 + \frac{3}{5} + \frac{3}{25} + \cdots + \frac{3}{5^n} = \dfrac{3 - \frac{3}{5^{n+1}}}{1 - \frac{1}{5}} = \frac{5}{4} \cdot 3 \cdot \left(1 - \frac{1}{5^{n+1}}\right)$.

We have $\lim\limits_{n\to\infty} P_n = \lim\limits_{n\to\infty} \frac{15}{4}\left(1 - \frac{1}{5^{n+1}}\right) = \frac{15}{4}\left(1 - \lim\limits_{n\to\infty} \frac{1}{5^{n+1}}\right) = \frac{15}{4}$. Since the sequence of partial sums converge to a limit $\left(\frac{15}{4}\right)$, we know the series converges (by definition) and that the sum, $\sum\limits_{n=0}^{\infty} \frac{3}{5^n} = \frac{15}{4}$.

(iii) Note that $E_n = \frac{15}{4} - P_n = \frac{15}{4} - \frac{15}{4}\left(1 - \frac{1}{5^{n+1}}\right) = \frac{15}{4(5^{n+1})}$.

Thus, $\lim\limits_{n\to\infty} E_n = \lim\limits_{n\to\infty} \frac{15}{4(5^{n+1})} = 0$. By the Vanishing Error Theorem, the series converges to $\frac{15}{4}$.

□ Exercises for Lesson IS 4.

For each of the following series in exercises 1 to 5,

(a) Make a table of decimal values of a_n and P_n for the first five n values, $n = 25$, and $n = 100$. For exercises 2 and 5, also tabulate values of E_n; for 2 use the conjecture that $L = \pi^4/90$ and for 5, make a conjecture about the value of L (hint: use IS 1, Exercise 1).

(b) Using your table in (a), estimate $\lim\limits_{n\to\infty} a_n$; what can you conclude about convergence of the series from this limit?

(c) Using your table in (a), estimate $\lim\limits_{n\to\infty} P_n$; what can you conclude about convergence of the series from this limit?

1. $\displaystyle\sum_{n=0}^{\infty} (-1)^n$ **2.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ **3.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n+1}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ **5.** $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$

For each of the following series in exercises 6 to 10,

(a) State the a_n formula and find the theoretical $\lim\limits_{n\to\infty} a_n$; what does this say about the series $\displaystyle\sum_{n=k}^{\infty} a_n$?

(b) Find a closed formula for P_n (expand to $+\cdots+$ and simplify to a formula without $+\cdots+$ or summation) and find the theoretical $\lim\limits_{n\to\infty} P_n$; what does this say about the series $\displaystyle\sum_{n=k}^{\infty} a_n$?

(c) (**7** and **8** only) Find a formula for E_n, simplify as much as possible, and find $\lim\limits_{n\to\infty} E_n$.

6. $\displaystyle\sum_{n=1}^{\infty} 1$ **7.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ hint: geometric **8.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

9. $\displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$ **10.** $\displaystyle\sum_{n=0}^{\infty} x^n$ for $x > 1$

■ Lesson IS 5: Special Series - Harmonic and Alternating Series

In this section, we concentrate on two particular series, $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ and their convergence or divergence. This study will lead to helpful principles about any alternating series and a way to compare infinite series of positive terms to an area under a curve.

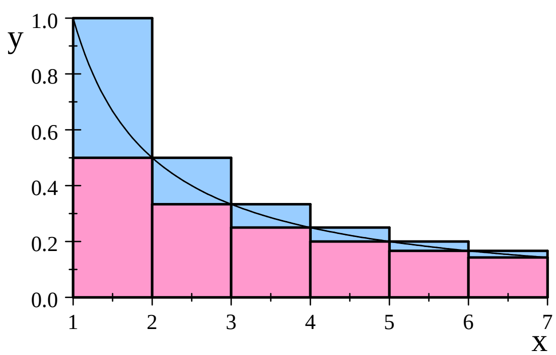
□ The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$ and the Integral Test.

The first step in determining the convergence properties of a series is to compute $\lim_{n\to\infty} a_n$ which in this case is $\lim_{n\to\infty} \frac{1}{n} = 0$. Following the warning after the Divergence Theorem, we note that this gives no information regarding the convergence of the series. Computing a few P_n values

$$P_5 = 2.2833,\ P_{25} = 3.816,\ P_{100} = 5.1874,\ \text{and}\ P_{200} = 5.878$$

hints that perhaps the series does not converge, but not very convincingly. Does P_n tend to infinity or do the values settle down to a limit after a while? Compute some more values of P_n for large values of n and try to determine the answer. Be careful: to be sure of your result, you have to use very large n and the computer has a tough time with 1/(a very large number). Let's use a different method.

We will compare the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ to the integral $\int_1^{\infty} \frac{1}{x}\,dx$. This integral represents the area under the curve $\frac{1}{x}$ for $x \geq 1$.



From the picture and using left and right hand sums, it is clear that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} \leq \int_1^7 \frac{1}{x}\,dx \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}.$$

More generally we have,

$$\sum_{n=2}^{N} \frac{1}{n} \leq \int_1^N \frac{1}{x}\,dx \leq \sum_{n=1}^{N-1} \frac{1}{n}. \tag{(1)}$$

We now let $N \to \infty$. Note that

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{N\to\infty} \int_1^N \frac{1}{x}\,dx = \lim_{N\to\infty} \ln x\Big|_1^N = \lim_{N\to\infty} \ln N = \infty.$$

So, the improper integral, $\int_1^\infty \frac{1}{x}\,dx$ diverges. That is, the area under the curve is infinite. From the inequality in (1) we see that

$$\lim_{N\to\infty} \int_1^N \frac{1}{x}\,dx \le \lim_{N\to\infty} \sum_{n=1}^{N-1} \frac{1}{n}$$

and so, $\lim_{N\to\infty} P_{N-1} = \lim_{N\to\infty} \sum_{n=1}^{N-1} \frac{1}{n} = \infty$. Thus the harmonic series diverges.

This example motivates the following "test" for the convergence or divergence of a series.

Theorem 23 (Integral Test) *Let $\sum_{n=k}^{\infty} a_n$ be a series of nonnegative terms, and let $f(x)$ be a function such that $f(n) = a_n$ for every natural number $n \ge k$ and f is decreasing and positive on the interval $[k, \infty)$. Then, $\sum_{n=k}^{\infty} a_n$ converges if and only if the improper integral $\int_k^\infty f(x)\,dx$ converges.*

See the next section for more discussion on this test. We now turn to a similar series, but one which has very different properties. This series provides motivation for the very useful Alternating Series Theorem.

□ **The Alternating Harmonic Series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ and the Alternating Series Theorem.**

The alternating signs in this series make it behave differently from the Harmonic Series. In fact, very differently – the Harmonic Series diverges and the Alternating Harmonic Series converges! To understand this, we compute a few values of P_n and make some observations:

$$P_1 = \sum_{n=1}^{1} \frac{(-1)^{n+1}}{n} = 1$$

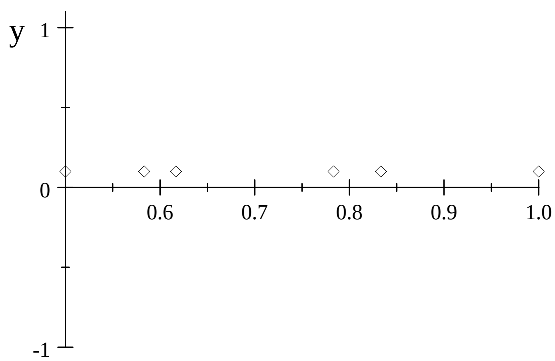
$$P_2 = \sum_{n=1}^{2} \frac{(-1)^{n+1}}{n} = \tfrac{1}{2} = .5$$

$$P_3 = \sum_{n=1}^{3} \frac{(-1)^{n+1}}{n} = \tfrac{5}{6} = .833$$

$$P_4 = \sum_{n=1}^{4} \frac{(-1)^{n+1}}{n} = \tfrac{7}{12} = .5833$$

$$P_5 = \sum_{n=1}^{5} \frac{(-1)^{n+1}}{n} = \tfrac{47}{60} = .7833$$

$$P_6 = \sum_{n=1}^{6} \frac{(-1)^{n+1}}{n} = \tfrac{37}{60} = .6167$$



Notice the "bracketing effect" that squeezes these partial sums together. That is, P_3 is between P_1 and P_2, P_4 is between P_2 and P_3, etc. In general, P_n is between P_{n-2} and P_{n-1}. This is due to the alternating signs of the terms and the fact that $|a_n| < |a_{n-1}|$. The limit of these partial sums (the sum of the infinite series) is squeezed to one finite number, since $\lim_{n\to\infty} a_n = 0$.

So, we know that the Alternating Harmonic series converges, the next obvious question to ask is: Can we find the sum and, if not, how closely can we approximate it? Since P_m is between P_n and P_{n+1} for every $m \ge n+2$, it must be that the infinite sum, $L = \lim_{n\to\infty} P_n$ is between P_n and P_{n+1}. In symbols, $|L - P_n| \le |P_{n+1} - P_n|$. Now the difference $P_{n+1} - P_n = a_{n+1}$. So, $|L - P_n| \le |a_{n+1}|$, meaning that the actual sum L differs from the approximation P_n by at most $|a_{n+1}|$. We summarize in the following,

Theorem 24 (Alternating Series.) *If the sequence b_n decreases to a limit of zero (denoted $b_n \searrow 0$), then the alternating series* $\displaystyle\sum_{n=k}^{\infty}(-1)^n b_n$ *converges to some limit L. Moreover,* $|E_n| = |L - P_n| \leq b_{n+1}$.

In other words, an error bound for approximating a convergent alternating series is given by the next term (that is the first term you leave off the approximation). The same theorem holds for $\displaystyle\sum_{n=k}^{\infty}(-1)^{n+1}b_n$

□ **Use of the error bound.**

To help understand the use of the error bound, we employ common English usage of $\pm$ in phrases such as "50 people plus or minus 10." This notation is often used in elementary science to express an experimental quantity, such as $q = 7.35 \pm .02$ for $7.33 \leq q \leq 7.37$.

We are in a very common situation: we know that our series converges but we do not know the exact limit L. We can compute a partial sum P_n but the exact error E_n is defined in terms of the unknown L, $E_n = L - P_n$. So, the equation $L = P_n + E_n$ is not very helpful. However, we often know an upper bound for how large E_n can be in absolute value. Say we know $|E_n| \leq M$, so that M is the error bound. Then we could write

$$L = P_n \pm M.$$

Here's why we can do this:

$$\begin{aligned}
\text{By definition, } |E_n| &= |L - P_n|\,. \\
\text{So, } -M &\leq L - P_n \leq M. \\
\text{Thus, } P_n - M &\leq L \leq P_n + M. \\
\text{or, using } \pm \text{ notation, } L &= P_n \pm M.
\end{aligned}$$

Returning to the Alternating Harmonic Series, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = L$. But, what is L? It's hard to say.

We can compute $P_{99} = .698172$ with error bound given by the alternating series theorem to be the next term, not including sign, of $1/100$. So, $L = .698172 \pm .01$. Going further out in the series, we have $P_{9999} = .693197$ with error bound $1/10000$. So, $L = .693198 \pm .0001$. Often, there is no better way to determine the infinite sum. We should round our approximations to significant digits agreeing with the error bound. Technically we should also round up the error bound, but let's not get carried away. It is a good estimate of the relative accuracy. If you are intrigued by this particular example, consider the series for $\ln(1 + x)$, [HG, p. 604] .

Remark 25 (WARNING!) *This convenient use of the "next term error bound" only works for ALTERNATING SERIES. For other series, it is false.*

□ **Positive Terms vs. Alternating Signs (Absolute Convergence).**

This is worth memorizing:

The Harmonic Series Diverges and The Alternating Harmonic Series Converges

If you have the impression that it is easier for a series with alternating signs to converge than for a series of positive terms, you are RIGHT! Either alternation $(-1)^n$ or $(-1)^{n+1}$ works the same, but we'll just state the former and assume the latter by analogy.

Consider the behavior of the positive terms b_n.

- If $\lim_{n\to\infty} b_n \neq 0$, then its also true that $\lim_{n\to\infty} (-1)^n b_n \neq 0$, so

  $\sum_{n=k}^{\infty} (-1)^n b_n$ diverges by the Divergence Test and

  $\sum_{n=k}^{\infty} b_n$ diverges by the Divergence Test.

- If $\lim_{n\to\infty} b_n = 0$, then as long as each b_n is smaller than the previous one,

  $\sum_{n=k}^{\infty} (-1)^n b_n$ converges by the Alternating Series Theorem, but

  $\sum_{n=k}^{\infty} b_n$ may or may not converge (inconclusive tests).

It is in some sense "harder" for series of positive terms to converge. The Absolute Convergence Theorem states that if $\sum_{n=k}^{\infty} b_n$ converges then $\sum_{n=k}^{\infty} (-1)^n b_n$ (or inserting any signs whatsoever) must also converge. If you know that $\sum_{n=k}^{\infty} b_n$ converges, then you can say that $\sum_{n=k}^{\infty} (-1)^n b_n$ *converges absolutely*, to communicate that not only the alternating series converges, but that the stronger statement holds, namely that the series converges even if you change every term to its absolute value. This is an important relationship but the terminology can be confusing. Remember that it is easier for an alternating series to converge.

☐ Exercises for Lesson IS 5

1. Show that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by comparing it to an area given by an improper integral. Show the area sketch, including an area corresponding to the infinite series. Evaluate the improper integral.

2. The series $\sum_{n=0}^{\infty} (-1)^n \, 2^{-n}$ is both a geometric series and an alternating series.

   (a) Calculate P_7 to four decimal places.

   (b) Using the alternating series theorem, bound $|E_7|$.

   (c) Use the geometric series formula find the exact value of L, then calculate $E_7 = L - P_7$ to four decimal places.

For problems 3 to 6, decide if the series converges or diverges by referencing a theorem. If it converges, use the given m to calculate P_m, bound $|E_m|$, and express $L = P_m \pm$error bound.

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $\quad m = 99.$

4. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ $\quad m = 10.$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n \, n}{n+1}$ $\quad m = 20.$

6. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \, n}{n^2+1}$ $\quad m = 20.$

7. Compute $\sum_{n=1}^{\infty} \frac{(-1)^n \, n}{(2n+1)!}$ to four decimal places by making a table of a_n and P_n values starting at $n = 1$ and stopping as soon as the alternating series theorem guarantees the result accurate to four decimal places.

8. For $\sum_{n=1}^{\infty} \frac{1}{n^5}$, the error $E_n = \left(\sum_{i=1}^{\infty} \frac{1}{i^5} \right) - P_n = \sum_{i=n+1}^{\infty} \frac{1}{i^5}$.

   (a) Use an area diagram to show that this last sum is LESS than $\int_n^{\infty} \frac{1}{x^5} dx$. Don't try to draw to scale, just start at $x = n$ and show $1/x^5$ as a decreasing function. Hint: the first number (height) in the sum is $n + 1$.

   (b) Compute $\int_n^{\infty} \frac{1}{x^5} dx$.

   (c) By (a), (b) and the Vanishing Error Theorem, show the series converges.

   (d) Using (a) and (b), express $L = P_{20} \pm$ error bound.

   (e) Use the same method to estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ with error less than 10^{-6}.

Exercises for Lesson IS 5 (continued)

9. Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges. (No need to draw areas here, just reference the theorem.)

10. Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p >$____. (Fill in the blank!)

11. Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $0 < p < 1$. (So $-p + 1 > 0$)

■ Lesson IS 6: Comparison Tests for Series

We have examined the following series in various degrees of detail:

- The Alternating Harmonic Series - $\sum \frac{(-1)^{n+1}}{n}$, which converges.
- Exponential - $\sum \frac{x^n}{n!}$ (see problem 5 in Lesson IS 4 homework), which converges for all x.
- Geometric Series - $\sum ax^n$, which converges if $|x| < 1$.
- Harmonic Series - $\sum \frac{1}{n}$, which diverges. (Note that this one is a special case of the p-series.)

We now wish to use these series and two comparison tests to determine whether or not a given series converges or diverges. The comparison tests work only for series with positive terms. Therefore, in this section, we concentrate on the last three series on the above list, with $x > 0$ in the exponential and geometric series. First, we generalize the results in the last section to series of the form, $\sum \frac{1}{n^p}$ for p any real number. Most of this was completed in the homework and for completeness, we state the theorem and the proof:

Theorem 26 (p-series) *The series* $\sum \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

Proof. First, note that if $p \leq 0$, then the terms of the series are not approaching 0 and so by the Divergence Test, the series diverges. Next, note that if $p > 0$ then $f(x) = 1/x^p$ is a nonnegative, decreasing function and $f(n) = 1/n^p$. We can apply the Integral Test:

$$\begin{aligned} \int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^p}\,dx \\ &= \lim_{b\to\infty} \left(\frac{b^{-p+1} - 1}{-p+1} \right) \qquad \text{if } p \neq 1 \\ &= \begin{cases} \frac{1}{p-1} \text{ if } p > 1 \\ \text{diverges if } 0 < p < 1 \end{cases} \end{aligned}$$

Combining this with the Harmonic series, the proof of the theorem is complete. ■

We now give two theorems which enable us to determine the convergence properties of many series of non-negative terms. If a series has some negative terms, it is possible for the partial sums to oscillate and so that the series diverges, as in $\sum (-1)^n$. However, with only non-negative terms, the sequence of partial sums is always increasing (or equal). If increasing partial sums are not bounded above, the series diverges to infinity. If increasing partial sums are bounded above, then a real analysis theorem says that they must converge to some limit, so the series converges. We just need to find some comparison to show that the series is bounded (finite) or not (infinite).

Theorem 27 (Comparison Test) *(a) If* $0 \leq a_n \leq b_n$ *for all* $n \geq 1$, *then* $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$,

so if $\sum_{n=1}^{\infty} b_n$ *converges (to a finite value), then* $\sum_{n=1}^{\infty} a_n$ *converges.*

(b) If $0 \leq b_n \leq a_n$ *for all* $n \geq 1$ *and* $\sum_{n=1}^{\infty} b_n$ *diverges (to infinity),then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

In other words, if we think a series converges, we just need to show it is smaller (term by term) than a series which we know converges. And, if we think a series diverges, we need to show it is larger (term by term) to a known divergent series. Sometimes, the terms of the two series being compared don't have the needed relationship for small n. But the convergence or divergence of a series depends only on all terms past m for some $m \geq 1$. Preceding terms definitely affect the actual sum but not the matter of whether the sum diverges to infinity or not. This gives us the modified comparison test.

Theorem 28 (Modified Comparison Test) *For any positive integers m, j, k.*

(a) If $0 \leq a_n \leq b_n$ for all $n \geq m$, then $\displaystyle\sum_{n=m}^{\infty} a_n \leq \sum_{n=m}^{\infty} b_n$,

so if $\displaystyle\sum_{n=j}^{\infty} b_n$ converges (to a finite value), then $\displaystyle\sum_{n=k}^{\infty} a_n$ converges.

(b) If $0 \leq b_n \leq a_n$ for all $n \geq m$ and $\displaystyle\sum_{n=j}^{\infty} b_n$ diverges, then $\displaystyle\sum_{n=k}^{\infty} a_n$ diverges.

This theorem is more general and hence more useful. This next theorem is useful when it is difficult or not possible to compare two series term by term, yet we suspect that the given series behaves like a series whose convergence (or not) we know.

Theorem 29 (Limit Comparison Test) *Let $\displaystyle\sum_{n=k}^{\infty} a_n$ and $\displaystyle\sum_{n=k}^{\infty} b_n$ be non-negative series. Suppose $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is a positive number (not zero and not infinity).*

(a) If $\displaystyle\sum_{n=k}^{\infty} b_n$ converges then $\displaystyle\sum_{n=k}^{\infty} a_n$ converges.

(b) If $\displaystyle\sum_{n=k}^{\infty} b_n$ diverges then $\displaystyle\sum_{n=k}^{\infty} a_n$ diverges.

This theorem says that if the terms of two series are eventually proportional to each other, then the convergence behavior of the two series is the same. Why is this true? Since $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$, for any $\varepsilon > 0$, $L - \varepsilon < \dfrac{a_n}{b_n} < L + \varepsilon$ for sufficiently large n. Thus, $(L-\varepsilon)\, b_n < a_n < (L+\varepsilon)\, b_n$. By the modified comparison test, if $\sum b_n$ converges, then $\sum a_n$ converges ($\sum a_n < (L+\varepsilon) \sum b_n$). If $\sum b_n$ diverges, then $\sum a_n$ diverges.

Note that L is a positive real number, not ∞ and not 0. Also note that both the Comparison Test and the Limit Comparison Test are used only for series with positive terms. Now we look at a few examples:

Example 30 *Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+1}$ converge or diverge?*

This looks like it should behave the same as $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges (p-series, $p = 2$). Use the comparison test. Since $2n^2+1 \geq n^2$ for $n \geq 1$, $\dfrac{1}{2n^2+1} \leq \dfrac{1}{n^2}$. Thus $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is finite, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2+1}$ converges by the comparison test.

Example 31 *Does $\displaystyle\sum_{n=2}^{\infty} \frac{2n+1}{3n^2-7}$ converge or diverge?*

We can eliminate the lower order constant terms since, for every $n \geq 2$,

$$\frac{2n+1}{3n^2-7} \geq \frac{2n}{3n^2-7} \geq \frac{2n}{3n^2} = \frac{2}{3n}.$$

The series $\displaystyle\sum_{n=2}^{\infty} \frac{2}{3n}$ diverges since it is a constant multiple of the harmonic series. (Specifically, $\displaystyle\sum_{n=2}^{\infty} \frac{2}{3n} = \frac{2}{3}\sum_{n=2}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Although arithmetic with divergent series can lead to invalid conclusions, this principle of a constant multiple of known series is valid since the partial sums, and hence limits, are simply constant multiples.) Thus, $\displaystyle\sum_{n=2}^{\infty} \frac{2n+1}{3n^2-7}$ diverges by comparison.

Example 32 *Does* $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{3n^2+7}$ *converge or diverge?*

Attempt (1): The straight-forward comparison, as in the previous example, fails. Observe for every $n \geq 1$,

$$\frac{2n-1}{3n^2+7} \leq \frac{2n}{3n^2+7} \leq \frac{2n}{3n^2} = \frac{2}{3n}.$$

But $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3n}$ diverges, so we only see that $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{3n^2+7} \leq \infty$, which doesn't lead to any conclusion. Still, it "should" behave like the harmonic series, calling for the limit comparison test.

Attempt (2): For limit comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, evaluate

$$\lim_{n\to\infty} \left(\frac{\frac{2n-1}{3n^2+7}}{\frac{1}{n}} \right) = \lim_{n\to\infty} \left(\frac{2n-1}{3n^2+7} \cdot \frac{n}{1} \right) = \lim_{n\to\infty} \frac{2n^2-n}{3n^2+7} = \lim_{n\to\infty} \frac{2-1/n}{3+7/n^2} = \frac{2}{3}.$$

Since this is a non-zero finite number, and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the limit comparison test tells us that $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{3n^2+7}$ also diverges.

Attempt (3): Actually, it is possible to use the regular comparison test but, since we are looking for divergence, we must go down in size. We also need to keep the dominate powers which may be accomplished by

$$\frac{2n-1}{3n^2+7} \geq \frac{2n-n}{3n^2+7} \geq \frac{2n-n}{3n^2+n^2} = \frac{1}{4n}$$

which is valid for $n \geq 3$. Since $\displaystyle\sum \frac{1}{4n}$ diverges as a mulitple of the hamonic series, then $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{3n^2+7}$ diverges by the (modified) comparison test.

Attempts (2) and (3) are both fine solutions; it is just a matter of personal preference and insight which you use.

Example 33 *Does* $\displaystyle\sum \frac{n}{2^n}$ *converge or diverge?*

Note that 2^n dominates n. That is, for every $n > 0$, $2^n > n$. Thus, we may suspect that in the long run (which is all that matters), the series behaves like $\displaystyle\sum \frac{1}{2^n}$, which converges since its a geometric series. A limit comparison of these two would be inconclusive, since their ratio is n which goes to infinity. However, we can do a regular comparison if we go up and try to get the dominate form x^n.

Note that if $x > 1$ and n is large enough, then $n < x^n$. If in addition, $x < 2$, then $\displaystyle\sum \frac{x^n}{2^n}$ converges (its geometric!). So, if we let $x = 1.5$ then we have, $\displaystyle\frac{n}{2^n} \leq \frac{(1.5)^n}{2^n}$. Since $\displaystyle\sum \frac{(1.5)^n}{2^n} = \sum \left(\frac{1.5}{2}\right)^n$ converges by geometric series with $1.5/2 = 3/4 < 1$, we have that $\displaystyle\sum \frac{n}{2^n}$ converges by the comparison test,. Note that we did NOT compare the series $\sum \frac{n}{2^n}$ with the series $\sum \frac{1}{2^n}$ in the proof, it was only used to help make an initial guess about the convergence.

☐ **The following tips may be useful when using comparison tests:**

1. See what dominates (is eventually larger) in the expression, a_n. Use the following facts:

$$\text{For fixed } p > 0 \text{ and } b > 1,$$
$$\ln n \le n^p \le b^n \le n! \quad \text{for large enough } n.$$

   This can be thought of as four different types or classes of functions, with a constant being below all of these. For example, $\ln n \le \sqrt[3]{n}$ for large enough n (can you verify this?). Within a class, the higher number dominates, $n^2 \le n^3$ and $2^n \le 3^n$.

   What dominates the numerator? What dominates the denominator? If these are different classes, then the smaller class function is relatively insignificant. For same class numerator and denominator, the ratio matters.

2. Decide by above whether you predict convergence or divergence. For convergence, go up overall, which means down in the denominator. For divergence go down, which means up in the denominator.

3. Usually, you want to see what class of function dominates overall and change to just one of this class in both numerator and denominator.

4. Never replace expressions in both the numerator and denominator in the same step, since it can be hard to detemine the overall effect.

□ Homework IS 6.

Determine whether the given series converge or diverge. Be sure to state any tests used and justify all steps.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4 + 5n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n+1)^3}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{7^n}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{5n^2}{n^3 + 1}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n} - 1}{n^2 + n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n + 3^n}$

9. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^9}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n^2 + 3^n}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n! + n}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+1)! + 7}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^{n^2}}$

16. Use the integral test to determine those values of p for which the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges.

■ Lesson IS 7: Convergence of Taylor Series

For x such that the series converges, we can write,

$$f(x) \quad = \quad f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \quad = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

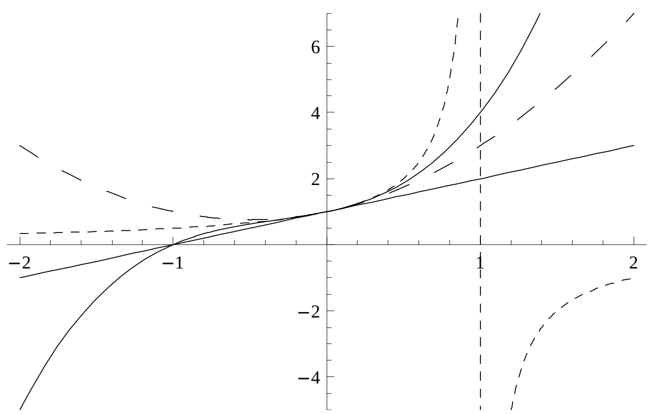
This series is called the *Taylor Series* of f about a (compare with the Taylor Polynomial of IS 1). If $a = 0$, then we call it the *Maclaurin Series*. The question becomes for which x does the series converge and does it really sum to the function f? We will see that this is true for some *interval of convergence* for x, that turns out symmetric around a. The interval depends on the function. We will graphically estimate the interval of convergence and learn to prove convergence using Taylor's Theorem from IS 1.

Example 34 *Find the Maclaurin Series for $f(x) = \frac{1}{1-x}$ and graphically estimate its interval of convergence.*

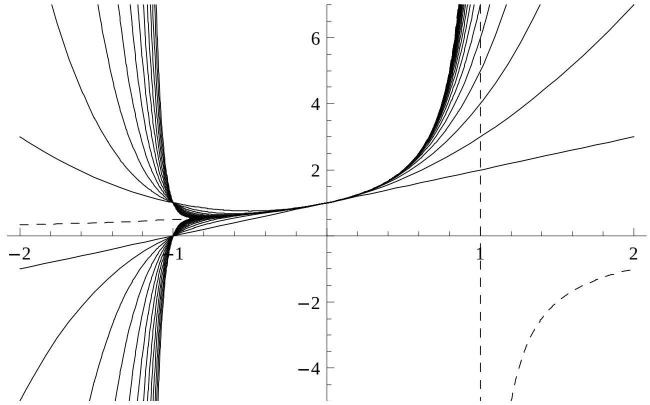
The reader should compute several derivatives in $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$ to see that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$. We could now plot the function together with progressively higher-order Taylor polynomials to estimate where the "infinite polynomial" might match the function. Actually, Mathematica can automatically compute the series of whatever expression you want and plot several Taylor polynomials using the following program. To enter this into Mathematica, type three underscore characters after "options", so that beginners can ignore this but advanced users may add graphics options.

```
taylorPlots[expr_, a_, n_, {x_, xmin_, xmax_}, {ymin_, ymax_}, options___] :=
  Module[{i, polyList},
   polyList = Table[Normal[Series[expr, {x, a, i}]], {i, 1, n}];
   Plot[{expr, polyList}, {x, xmin, xmax}, PlotRange -> {ymin, ymax}, options]
   ];
```

So `taylorPlots[1/(1-x), 0., 3, {x, -2, 2}, {-5,7}]` will plot the first 3 Maclaurin polynomials (starting with $p_1(x)$) using coloring, instead of the dashed original function and long-dashed quadratic below.



It appears that the linear polynomial "fits" around (-.2,.2), while the quadratic and cubic fit closely on (-.4,.4). What if we were to go to high order, even infinity? How big would the interval of convergence be where the infinite series could match the function? We get a better sense of this by going up to the 25th order polynomial, by replacing 3 with 25 in the `taylorPlots` command.



It appears that the higher-orders hug up next to the function on the interval (-1,1), and "explode" (diverge) outside this interval. If we were able to sum all the way to infinity, the infinite series would be the graph of $\frac{1}{1-x}$ on the interval (-1,1) and simply not exist outside this interval. This interval of convergence is the domain of the series function. So we estimate that $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ for $-1 < x < 1$.

Of course, this is an estimate, could we prove it? Yes, we already proved this in IS 2 on geometric series. This geometric series has a simple formula for the partial sums which converge to $\frac{1}{1-x}$ if and only if $|x| < 1$. For functions whose series are not geometric, it is usually not possible to get a useful formula for the partial sums. However, Taylor's Theorem gives a formula for the error in the partial sum approximation. Thus we must turn to the Vanishing Error Theorem to prove convergence for other functions. We could prove geometric series convergence by this theory as well.

Before considering this theory, here are a few hints on using `taylorPlots`. Don't forget the correct `a`, start with smaller `n` and experiment with `xmin, xmax, ymin, ymax` to get a good scale. In Mathematica 6, you may want to use a variable `n`, by wrapping `Manipulate[____,{n,1,25,1}]` around the `taylorPlots` command, e.g., fill in these blanks with `taylorPlots[1/(1-x), 0., n, {x, -2, 2}, {-5,7}]`. On the resulting graphic, click + on the slider bar and then click the + button to step through polynomials.

A rigorous way to show that a Taylor series converges to $f(x)$ on an interval, say $[-d, d]$, is to use Taylor's Theorem and the Vanishing Error Theorem for x values in the interval. As long as we can get an estimate of the maximum $\left|f^{(n+1)}(x)\right|$ on $[-d, d]$, the problem can boil down to the known limit $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$.

Example 35 *Show that for every d, e^{2x} is equal to its Taylor Series on the interval $[-d, d]$. Since d is arbitrary, this shows that the interval of convergence is the whole real line.*

First, the Taylor Series for $e^{2x} = \displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$ as can be derived by computing derivatives and observing the pattern. Using Taylor's Theorem (see IS 1) for each n, we have that

$$|f(x) - p_n(x)| \quad = \quad |E_n(x)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}|x|^{n+1} \text{ for some } c \text{ between } x \text{ and } 0.$$

The plan of the proof is to take the limit as $n \to \infty$ and, as long as the derivative does not grow too big, the factorial will dominate and the limit will be zero; then Vanishing Error shows that the series converges to $f(x)$. Sometimes, it is easy to bound the derivative by a constant M at any point, as in the sin or cos functions. In general, as n varies toward infinity, x is constant but actually c is not; c depends on n and must be replaced by a constant. In our example, the derivative also depends on n so we have to be careful.

Note that the n^{th} derivative of $f(x) = e^{2x}$ is $2^n e^{2x}$. From the graph of e^{2x} on the interval $[-d, d]$, we see that the maximum value of e^{2x} occurs at $x = d$. Also, c must be in the interval $[-d, d]$, since x is. So we conclude that $e^{2c} \leq e^{2d}$. Thus we have,

$$
\begin{aligned}
|E_n(x)| &= \frac{|f^{(n+1)}(c)|}{(n+1)!}|x|^{n+1} = \frac{2^{n+1}e^{2c}}{(n+1)!}|x|^{n+1} \\
&\leq \frac{2^{n+1}e^{2d}}{(n+1)!}|x|^{n+1} \text{ for every } x \text{ in } [-d,d].
\end{aligned}
$$

Now, we group powers and have,

$$
|E_n(x)| \leq \frac{(2\,|x|)^{n+1}e^{2d}}{(n+1)!} = e^{2d}\frac{(2\,|x|)^{n+1}}{(n+1)!}
$$

We now use the Vanishing Error Theorem to complete the proof.

$$
\begin{aligned}
\lim_{n\to\infty}|E_n(x)| &\leq \lim_{n\to\infty} e^{2d}\frac{(2\,|x|)^{n+1}}{(n+1)!} \\
&= e^{2d}\lim_{n\to\infty}\frac{(2\,|x|)^{n+1}}{(n+1)!} \\
&= 0 \;\text{ by the fact right before the example } (\lim_{n\to\infty}\frac{x^n}{n!} = 0).
\end{aligned}
$$

Thus, for all x in $[-d,d]$, the partial sums (Taylor polynomials) converge to $f(x)$. Since d is arbitrary, $\displaystyle\sum_{n=0}^{\infty}\frac{2^n x^n}{n!} = e^{2x}$ for all real x.

One of the goals in this section is to directly use derivatives and such arguments to establish that each of the following functions is equal to a specific series on a specific interval of convergence:

$$
e^x, \quad \sin x, \quad \cos x, \quad \frac{1}{1-x}, \quad (1+x)^p.
$$

These results will be our "library of series." We may establish the series for $\sin x$ in class (take notes!) and $\dfrac{1}{1-x}$ is done above; the others are left as homework. Once this library of series is established, we will rarely need to rely on the definition of Taylor and Maclaurin series (using derivatives) to find series. How to do this will be discussed in the next section.

Actually, the equality of the function and the series can be used the other way around. We can use series to find derivatives! This is a very practical idea, known as automatic or computational differentiation, that is a very different way of thinking about differentiation (even to most mathematicians). Here is an introduction to this idea. Suppose we find a series for $f(x)$ about a, then it must be the Taylor series. This means that $\frac{f^{(n)}(a)}{n!}$ is the coefficient of the n^{th} power of $(x-a)$. In other words, if you want to know the n^{th} derivative of f evaluated at a, (i.e. $f^{(n)}(a)$), just multiply the coefficient of the n^{th} power of $(x-a)$ in the series for f by $n!$. An example is in order.

Example 36 *Find the 4^{th} and 21^{st} derivatives of $f(x) = \sin(2x)$ evaluated at 0.*

We first find the Taylor series for $\sin(2x)$. Substituting $2x$ for x in the library series for $\sin x$, we find

$$\begin{aligned} \sin(2x) &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1} \\ &= 2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 - \frac{128}{7!}x^7 + \cdots \end{aligned}$$

To find the 21^{st} derivative of $\sin(2x)$ evaluated at 0, we use the discussion preceding the example. The Taylor series formula for the coefficient of x^{21} is $\dfrac{f^{(21)}(0)}{21!}$. But, we know from the series that the coefficient of x^{21} is $\dfrac{(-1)^{10}2^{21}}{(21)!}$ (using $n = 10$). Equating these two coefficients of x^{21} we have that $f^{21}(0) = 21!\left(\dfrac{(-1)^{10}2^{21}}{(21)!}\right) = 2^{21} = 2.0972 \times 10^6$.

Note that there are no even powers of x in the series for $\sin(2x)$, so any even derivative of $f(x) = \sin(2x)$ evaluated at 0 is zero. In particular, $f^{(4)}(0) = 0$.

■ Exercises for Lesson IS 7.

1. Use derivatives to find the Maclaurin Series for e^{3x} in expanded and summation form.

2. Consider the function $f(x) = (1+x)^p$ for any constant p.

   (a) Use derivatives to find the Maclaurin Series in expanded form, showing at least four nonzero terms.

   (b) Simplify this in the specific cases $p = -3$ and $p = 1/2$.

   (c) Use the taylorPlots function to estimate the interval of convergence for $(1+x)^{-3}$.

3. Consider the function $f(x) = \ln(x/2)$.

   (a) Use derivatives to find the Taylor Series about $a = 2$ in expanded form and then try to find the summation notation. (caution: you should get $f'(x) = 1/x$)

   (b) Use the taylorPlots function to estimate the interval of convergence for this series. (In Mathematica, $\ln(x/2)$ is $Log[x/2]$.)

4. Consider the function $f(x) = \cos(x)$.

   (a) Use derivatives to find the Maclaurin Series. Specifically, find the pattern, show the expanded form and write in summation notation.

   (b) Prove that for any x, $\cos(x)$ is equal to its Taylor series by using Taylor's Theorem and the Vanishing Error Theorem. (So the interval of convergence is the whole real line.)

5. Consider the function $f(x) = e^x$.

   (a) Use derivatives to find the Maclaurin Series. Specifically, find the pattern, show the expanded form and write in summation notation.

   (b) Prove that for every $|x| < d$, e^x is equal to its Maclaurin Series, by using Taylor's Theorem and the Vanishing Error Theorem. (Since d is an arbitrary positive constant, the interval of convergence is the whole real line.)

6. In the next lesson, we see that $f(x) = x^2 \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+4}$. What is the coefficient of x^{16}? What is the 16^{th} derivative of f at 0?

■ Lesson IS 8: Manipulating Known Taylor Series

We now have the following "Library of Series":

$$\text{For } -\infty < x < \infty, \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\text{For } -\infty < x < \infty, \qquad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\text{For } -\infty < x < \infty, \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\text{For } -1 < x < 1, \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\text{For } -1 < x < 1, \qquad (1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \cdots$$

There are many ways to manipulate known series.

1. Substitute an expression for x in a series (AND in the interval of convergence!)
2. Substitute a series for a function $f(x)$ in an expression.
3. Differentiate the series (term by term). The interval of convergence for the resulting series is the same as the original series.
4. Integrate the series (term by term). The interval of convergence for the resulting series is the same as the original series. There is (of course) a constant of integration, best written at the front of the series.

The last two methods are actually theorems. Here are some examples to illustrate the above methods.

Example 37 *Write the series for $f(x) = e^{-2x}$. Give the first 4 terms and the summation form.*

Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

then

$$\begin{aligned} e^{-2x} &= 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \cdots \\ &= 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} x^n. \end{aligned}$$

Example 38 *Find the Maclaurin series in expanded form for the function $f(x) = \arctan x$. Give the interval of convergence.*

Note that $\arctan x = \displaystyle\int \frac{1}{1+x^2}dx$. So, we write the Taylor Series for $\dfrac{1}{1+x^2}$ and integrate.

$$\begin{aligned}
\int \frac{1}{1+x^2}dx &= \int \frac{1}{1-(-x^2)}dx \text{ now use the Maclaurin series for } \frac{1}{1-x} \\
&= \int (1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \cdots)dx \\
&= \int (1 - x^2 + x^4 - x^6 + x^8 + \cdots)dx \\
&= c + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots
\end{aligned}$$

We have

$$\arctan x = c + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots$$

We now find the constant of integration. Substitute $x = 0$ into both sides of the equation and solve for c :

$$\begin{aligned}
\arctan 0 &= c + 0 - \frac{1}{3}0^3 + \frac{1}{5}0^5 - \frac{1}{7}0^7 + \frac{1}{9}0^9 + \cdots \\
0 &= c
\end{aligned}$$

Thus, the series for $\arctan x$ is

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Since the interval of convergence for the series for $\dfrac{1}{1-x}$ is $|x| < 1$, the interval of convergence for $\dfrac{1}{1+x^2}$ is $|-x^2| < 1$ (which is still $|x| < 1$). Finally, then, the interval of convergence for $\arctan x$ is also $|x| < 1$.

See the homework for more examples. The following is a list of several applications of manipulating known Taylor series.

1. Find Taylor series for complicated functions - without taking derivatives.
2. Determine properties of a function. For example, even functions ($f(-x) = f(x)$) and odd functions ($f(-x) = -f(x)$) are easy to see with series.
3. Decide indeterminate forms in limits. For example, substituting series in a $0/0$ expression and cancelling gives a constant limit.
4. Find derivative values as in IS 7.
5. Solve differential equations as in IS 10 - IS 12.

□ Exercises for Lesson IS 8

Note - when showing the expanded form, write at least 4 nonzero terms and $+\cdots$.

1. Take the derivative of the series for $\sin x$ to get the series for $\cos x$ in expanded and summation form.
2. Integrate the series for $\sin x$ to get the series for $\cos x$ in expanded and summation form.
3. Substitute $i\theta$ for x in the series for e^x in expanded form (show 6 terms) and then derive Euler's formula.
4. Consider the function $f(z) = \dfrac{1}{\sqrt{1-z^2}}$.
   (a) Find the expanded form for the Maclaurin series for f.
   (b) What is the value of the 6^{th} derivative of f evaluated at 0? (i.e. find $f^{(6)}(0)$)
5. Use the previous problem to find the expanded form of the Maclaurin series for $g(z) = \arcsin z$. Is $\arcsin z$ an even function, odd function or neither? Justify your answer by explaining how powers (nonzero terms) in the series determine how $\arcsin(-z)$ relates to $\arcsin(z)$.
6. Find the summation form of the series for $\dfrac{1}{1+9x^2}$ along with the resulting open interval of convergence.
7. Consider the function $f(x) = \dfrac{x}{e^{x^2}}$.
   (a) Find the summation form of the series for f along with the resulting open interval of convergence.
   (b) What is the value of $f^{(27)}(0)$? Of $f^{(28)}(0)$?
8. Find the expanded form of the Maclaurin series for $f(t) = e^t \cos t$, showing four nonzero terms.
9. Use the expanded form of series to evaluate the following limits (do not use L'hopital's rule):
   (a) $\displaystyle\lim_{h\to 0} \frac{e^h - 1 - h}{h^2}$
   (b) $\displaystyle\lim_{\theta\to 0} \frac{\theta - \sin\theta}{\theta^3}$
10. Use integration to find the series for $\ln(1+x)$ about 0.

■ Lesson IS 9: Radius of Convergence and Analytic Functions

A series of the form $\sum_{n=k}^{\infty} c_n (x-a)^n$ is called a **power series**. Often, $a = 0$. The power series defines a function

$$f(x) = \sum_{n=k}^{\infty} c_n (x-a)^n$$

whose domain is the set of x values for which the series converges. It is "easiest" for the series to converge at the center, $x = a$ and becomes more difficult as $|x-a|$ grows. In fact, the techniques of this section will show that there is always some ρ, called the **radius of convergence**, such that the series converges if $|x-a| < \rho$ and diverges if $|x-a| > \rho$. The open interval $(a-\rho,\ a+\rho)$ is defined to be the **interval of convergence** of the power series. These are different ways of describing the same interval of symmetric distance from a, according to the equivalences:

$$|x-a| < \rho \Longleftrightarrow -\rho < x-a < \rho \Longleftrightarrow a-\rho < x < a+\rho \Longleftrightarrow x \text{ is in } (a-\rho,\ a+\rho)\,.$$

We include the possibility that $\rho = \infty$ so that the interval of convergence is the whole real line. We have $\rho = 0$ if the series "never" converges, except at $x = a$, where it is one constant term .

The only difference between the domain of the power series function and the interval of convergence is that the series may or may not converge at the left or right endpoints of the interval. You could plug the endpoints values in specific examples to see different endpoint behaviors, but this will not particularly interest us. The interval of convergence shows the limits of the domain of the function.

Example 39 *For what values of x does $\sum_{n=0}^{\infty} \frac{1}{3^n}(x-1)^n$ converge?*

This is the geometric series, $\sum_{n=0}^{\infty} \left(\frac{x-1}{3}\right)^n$ which converges if and only if $\left|\frac{x-1}{3}\right| < 1$. Simplifying, we have $|x-1| < 3$, which says that the radius of convergence is 3 and the interval of convergence is $(-2, 4)$. This is exactly the domain of the function $f(x) = \sum_{n=0}^{\infty} \left(\frac{x-1}{3}\right)^n$.

Almost any power series can be compared to a geometric series, in the limit, by the Ratio Test. This will result in an interval of convergence. The ratio of consecutive terms is used to recover the geometric ratio in the limit. In the above example, the ratio of the $(n+1)$st to the nth term is $\dfrac{\frac{1}{3^{n+1}}(x-1)^{n+1}}{\frac{1}{3^n}(x-1)^n} = \dfrac{x-1}{3}$, so the series converges if this is less than one in absolute value. In general, we use the function notation $u_n(x)$ to denote any series term that depends on n and may (but doesn't have to) depend on x.

Theorem 40 (Ratio Test) *Suppose $\lim_{n\to\infty} \left|\frac{u_{n+1}(x)}{u_n(x)}\right| = L(x)$. If $L(x) < 1$, then $\sum_{n=k}^{\infty} u_n(x)$ converges. If $L(x) > 1$, then $\sum_{n=k}^{\infty} u_n(x)$ diverges. (The Ratio Test is inconclusive if $L(x) = 1$.)*

Proof Sketch. For this rough explanation, let's assume the series terms $u_n(x)$ are positive numbers, and that $\lim_{n\to\infty} \frac{u_{n+1}(x)}{u_n(x)} = L(x)$, a real number with $0 < L(x) < \infty$. This means that for large n, say $n \geq M$, $\frac{u_{n+1}(x)}{u_n(x)} \approx L(x)$ or $u_{n+1}(x) \approx L(x) \cdot u_n(x)$. So,

$$\begin{aligned}
u_{M+1}(x) &\approx (L(x))\, u_M(x) \quad \text{and} \\
u_{M+2}(x) &\approx (L(x))\, u_{M+1}(x) \approx (L(x))^2\, u_M(x) \quad \text{and} \\
u_{M+3}(x) &\approx (L(x))\, u_{M+2}(x) \approx (L(x))^3\, u_M(x), \ \text{etc.}
\end{aligned}$$

The series beginning at M, looks like:

$$\begin{aligned}\sum_{n=M}^{\infty} u_n(x) &= u_M(x) + u_{M+1}(x) + u_{M+2}(x) + u_{M+3}(x) + \cdots \\ &\approx u_M(x) + (L(x))\, u_M(x) + (L(x))^2\, u_M(x) + (L(x))^3\, u_M(x) + \cdots \\ &= u_M(x)\left(1 + (L(x)) + (L(x))^2 + (L(x))^3 + \cdots\right).\end{aligned}$$

This last series is a geometric series and converges if and only if $L(x) < 1$. The approximate equality could be rigorously adapted to show that: if $L(x) < 1$, $\sum_{n=M}^{\infty} u_n(x) \leq$ a convergent geometric series so the series converges; or if $L(x) > 1$, $\lim_{n\to\infty} u_n(x) \neq 0$, so the series diverges. ■

Example 41 *Find the Taylor series about* 3 *and the interval of convergence for the function* $\ln(2x-5)$.

We can use derivatives to find the Taylor series about 3 for $\ln(2x-5)$:

$$\ln(2x-5) = 0 + 2(x-3) - \frac{2^2(x-3)^2}{2} + \frac{2^3(x-3)^3}{3} - \frac{2^4(x-3)^4}{4} + \cdots.$$

We could use taylorPlots to observe the radius of convergence, though this is not very precise. The Ratio Test nails the exact radius of convergence.

First we form the nth term formula to get

$$\ln(2x-5) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^n}{n}(x-3)^n.$$

The coefficient of the nth term is $u_n(x) = \dfrac{(-1)^{n+1}2^n(x-3)^n}{n}$. We use the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| &= \lim_{n\to\infty}\frac{\left(\frac{2^{n+1}|x-3|^{n+1}}{n+1}\right)}{\left(\frac{2^n|x-3|^n}{n}\right)} \\ &= \lim_{n\to\infty}\left(\frac{2^{n+1}|x-3|^{n+1}}{n+1}\right)\left(\frac{n}{2^n|x-3|^n}\right) = \\ &= \lim_{n\to\infty}\frac{2n}{n+1}|x-3| = 2|x-3|.\end{aligned}$$

So the series converges if $2|x-3| < 1$, or $|x-3| < 1/2$. The radius of convergence is $1/2$ and the interval of convergence is $(2.5, 3.5)$.

As long as $u_n(x)$ is a power series, the Ratio Test limit $L(x)$ will have a factor of the form $|x-a|$, so that $L(x) < 1$ results in $|x-a| < \rho$, giving the radius of convergence. In special cases, $L(x) = 0 < 1$ always, so $\rho = \infty$; on the other hand $L(x) = \infty$ is never less than 1, so $\rho = 0$. We commonly encounter series with only odd or only even terms, or maybe every third term as in the example below. In such cases, the radius of convergence is not quite as simple as the inverse of the constant in $L(x)$, so we encourage the standard ratio test, including setting the limit less than 1.

Example 42 *Find the interval of convergence for* $\sum_{n=1}^{\infty}\dfrac{x^{3n+1}}{n^2 5^n}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| &= \lim_{n\to\infty}\left(\frac{|x|^{3(n+1)+1}}{(n+1)^2\,5^{n+1}}\right)\left(\frac{n^2 5^n}{|x|^{3n+1}}\right) \\ &= \lim_{n\to\infty}\frac{|x|^3\,n^2}{(n+1)^2\,5} \\ &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2\frac{|x|^3}{5} = \frac{|x|^3}{5}.\end{aligned}$$

By the Ratio Test, the series converges if $\frac{|x|^3}{5} < 1$ or $|x| < \sqrt[3]{5}$. The radius of convergence is $\sqrt[3]{5}$ and the interval of convergence is $\left(-\sqrt[3]{5}, \sqrt[3]{5}\right)$.

Example 43 *Find the radius and interval of convergence of the power series* $\displaystyle\sum_{n=0}^{\infty} \frac{2(-1)^n x^n}{3^n n!}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| &= \lim_{n\to\infty}\left(\frac{2\,|x|^{n+1}}{3^{n+1}\,(n+1)!}\right)\left(\frac{3^n n!}{2\,|x|^n}\right)\\ &= \lim_{n\to\infty}\frac{|x|\,n!}{3(n+1)n!} = \lim_{n\to\infty}\frac{|x|}{3(n+1)} = 0.\end{aligned}$$

Since $0 < 1$ is always true, the series converges for all x by the Ratio Test. The interval of convergence is $\mathbb{R} = (-\infty, \infty)$ and the radius of convergence is ∞.

□ **Power Series as Functions**

By the last example, we know that $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{2(-1)^n x^n}{3^n n!}$ defines a function for all real numbers x. Does this function have an elementary formula that we would recognize? This is a kind of puzzle where you try to rewrite the series so that it can be recognized as something substituted into one to the series from our library. The best general hint is to group together all exponents of n. Lets try with this example:

$$f(x) = \sum_{n=0}^{\infty} \frac{2(-1)^n x^n}{3^n n!} = 2\sum_{n=0}^{\infty} \frac{(-\frac{x}{3})^n}{n!}.$$

Since the $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, we conclude that $f(x) = 2e^{-x/3}$.

If we decided to find the Taylor series (about 0) for $f(x) = 2e^{-x/3}$, what would result? The original power series, of course! Try it. The words power series and Taylor series are actually interchangeable. It's like the chicken or egg question. If we start with the series as in the example, we tend to call it a power series. If we start with the function and find its associated power series, we call that the Taylor series.

As another example, we find an elementary formula for the function described by the power series, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}(x-1)^n$ (see the first example). Using geometric series, we have

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{x-1}{3}\right)^n = \frac{1}{1-\dfrac{x-1}{3}} = \frac{3}{4-x}.$$

The power of power series is this: not every one can be written as an elementary formula. That is, A POWER SERIES CAN DESCRIBE A "BRAND-NEW" FUNCTION THAT WE'VE NEVER BEEN ABLE TO WRITE DOWN WITH A FORMULA BEFORE! In fact, Example 42 defines such a function $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^{3n+1}}{n^2 5^n}$ with domain $\left(-\sqrt[3]{5}, \sqrt[3]{5}\right)$. This is especially useful in differential equations (as we will soon see). Some very simple differential equations have as solutions functions which do not have an elementary formula.

□ **Power Series given by recurrence relations**

Example 44 *Let* $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ *where* c_0 *is a constant and* $c_{n+1} = \frac{-1}{n+1}c_n$. *Find (a) the radius of convergence, (b) a formula for* c_n*, and (c) an elementary formula for* $f(x)$.

(a) To use the Ratio Test, $\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}x^{n+1}}{c_nx^n}\right| = \lim_{n\to\infty}\left|\frac{\frac{-1}{n+1}c_nx^{n+1}}{c_nx^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 <$ 1, so $\rho = \infty$.

(b)
$$\begin{aligned}
c_1 &= (-1/1)c_0 = -c_0\\
c_2 &= (-1/2)c_1 = c_0/2\\
c_3 &= (-1/3)c_2 = -c_0/(3\cdot 2)\\
c_4 &= (-1/4)c_3 = c_0/(4\cdot 3\cdot 2)\\
&\vdots
\end{aligned}$$

We conclude that $c_n = \dfrac{(-1)^n c_0}{n!}$.

(c) $f(x) = \sum_{n=0}^{\infty} c_nx^n = \sum_{n=0}^{\infty}\frac{(-1)^n c_0}{n!}x^n = c_0\sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = c_0e^{-x}$.

□ Analytic Functions

The class of all functions that can be written as power series is very large and every such function is very nice. It is useful to define a name for this class of functions. Informally, we use the term analytic function to refer to any function that can be (or is) written as a power series with positive (non-zero) radius of convergence. Formally, we define analytic at a point:

Definition 45 *f is analytic at a means that f can be written as a power series about a with a positive radius of convergence.*

In symbols, $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for some c_n's such that $\rho > 0$.

Analytic functions have the following property: If f is analytic at the point a, then it must be that $c_n = \dfrac{f^{(n)}(a)}{n!}$ because of simple derivative rules. From this we have the following implications:

$$\begin{aligned}
f \text{ analytic at } a \quad &\Longrightarrow \quad f^{(n)}(a) \text{ exists for all } n\\
&\Longrightarrow \quad f \text{ is differentiable at } a\\
&\Longrightarrow \quad f \text{ is continuous at } a\\
&\Longrightarrow \quad f \text{ is defined in some open interval around } a.
\end{aligned}$$

The arrows above can not in general be reversed. This creates a spectrum of functions, where most fall to one extreme or the other, specifically, f is analytic at a or f is not defined at a. By using roots, absolute value, or piecewise defined functions, we have created examples in the Table below that fall in the intermediate levels, where the function has the stated property (and hence those properties below it) but not the property above it.

Property at 0	Examples (elementary formulas)	Piecewise defined functions
analytic at 0	$\frac{x^2\sin(x)}{e^x}$, $\frac{x}{(x+5)(x-2)}$, almost anything defined at 0	$f(x) = \left\{\begin{array}{c} \frac{\sin(x)}{x}, \text{ if } x \neq 0 \\ 1, \text{ if } x = 0 \end{array}\right\} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$
infinitely differentiable at 0		$f(x) = \left\{\begin{array}{c} e^{\left(-1/x^2\right)}, \text{ if } x \neq 0 \\ 0, \text{ if } x = 0 \end{array}\right\}$ has $\rho = 0$
$\vdots$	$\vdots$	$\vdots$
twice differentiable at 0	$x^{7/3}$	$f(x) = \begin{cases} x^3, \text{ if } x > 0 \\ 0, \text{ if } x \leq 0 \end{cases}$
differentiable at 0	$x^{4/3}$	$f(x) = \begin{cases} x^2, \text{ if } x > 0 \\ 0, \text{ if } x \leq 0 \end{cases}$
continuous at 0	$x^{1/3}$, $x^{2/3}$, $\lvert x\rvert$	$f(x) = \begin{cases} x, \text{ if } x > 0 \\ 0, \text{ if } x \leq 0 \end{cases}$
defined around 0		$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x \leq 0 \end{cases}$
defined at 0	$\sqrt{x}$	
a function on some domain	$\frac{x-2}{x(x+3)}$, $\ln(x)$, almost anything not defined at 0	

So, given an elementary formula and a point a, is the function it defines analytic at a? A good rule of thumb is yes, as long as the function is defined around a (in some open neighborhood of a) and does not use roots or absolute value. For example, $f(x) = \dfrac{e^x \sin(\ln(x))}{x-3}$ is analytic at every point $x > 0$ (the domain of $\ln x$) except at $x = 3$. That means f can be written as a power series (its Taylor series). However, it would be awfully tedious (at the very least) to compute by direct differentiation. Combining known series from the library is a better strategy, but still this one is best done by computer. (This is how the Mathematica Series command works.)

From the other point of view, given a power series, is the function it defines analytic? Yes, as long as the radius of convergence $\rho > 0$, the function is analytic within the interval of convergence. The question of whether it can be written as an elementary formula is not so easy. Sometimes it can, sometimes not.

□ Exercises for Lesson IS 9.

For each of the following power series, use the ratio test to find the radius of convergence and the (largest open) interval of convergence.

1. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n}(x-2)^n$ 2. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2}$ 3. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n\,4^n}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{n!}x^n$ 5. $\displaystyle\sum_{n=0}^{\infty} 9^n(x-1)^{2n}$ 6. $\displaystyle\sum_{n=0}^{\infty} n!x^n$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$ Hint: $\displaystyle\lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n = e$

8. Find elementary formulas for the analytic functions in 1 and 4.

For 9 and 10, $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ where the c_n'as are as prescribed. For each, find (a) the radius of convergence, (b) a formula for c_n, and (c) an elementary formula for $f(x)$.

9. $c_{n+1} = \frac{3}{4}c_n$ and c_0 is some arbitrary constant.
10. $c_{n+1} = \frac{7}{n+1}c_n$ and $c_0 = 5$.
11. Let $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ where the c_n'as are prescribed by $c_{n+2} = \frac{-3n}{n+1}c_n$. (a) If $c_0 = 0$ and $c_1 = 1$, find the coefficients through x^7 and write out the series in expanded $(+\cdots)$ form. (b) Since this series has only odd terms, use the ratio test $\displaystyle\lim_{n\to\infty}\left|\frac{c_{n+2}x^{n+2}}{c_n x^n}\right|$ to find the radius of convergence and interval of convergence.
12. Define a function where the third derivative is continuous in a neighborhood of 0 (open interval containing 0) but the fourth derivative is undefined at 0. Thus the function is not analytic at 0.

■ Lesson IS 10: [EP] 8.1 - Introduction and Review of Power Series.

This section covers the series solution method.

Suggested homework:

4, 7, 17, 21 (go ahead and use c_0 and c_1 values as soon as possible).

■ Lesson IS 11: [EP] 8.2 - Series Solutions Near Ordinary Points.

We start at the "Translated Series Solutions" subsection. Practice the method, using the substitution $t = x - a$ for initial conditions at a.

Suggested homework (for today, ignore instructions about the radius and interval of convergence):

14 (answer in expanded form), 18, 19, 21.

■ Lesson IS 12: [EP] 8.2 - Series Solutions Near Ordinary Points (Theory).

The discussion of analytic functions from Lesson IS 9 may be included and expanded here.

Suggested homework:

For each of the following problems, just state the form of the series solution about an ordinary point a (just fill in a, don't solve for recurrence) and justify the guaranteed radius of convergence using Theorem 1 for differential equations.

2, 5, 14, 18, 19, 20 (change initial conditions to $y(-1) = 2$ and $y'(-1) = 0$), 21.

Go through 8.2_SeriesSoln.nb

Suggested Homework:

Use Mathematica to solve 23 and 30.

■ Lesson IS 13: [EP] 8.3 - Regular Singular Points.

Suggested Homework:

1, 3, 5, 8, 17, 22, 38.

■ Odd numbered exercise (partial) solutions

□ IS 1

1. $p_n(x) = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n$
2.
3. (a) $p_3(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3$
   (c) $p_3(1.4) = .696$
   (d) $|E_3(1.4)| \leq .044$
4.
5. $p_3(x) = 5 - 3x + 7x^2 = p_{10}(x)$
6.
7. $c = f''(0)/2! =$concavity$/2 < 0$ since concave down.
8.
9. (b) about 0.125 or 0.06 with a tighter bound on the derivative in the interval.
   (d) $0.125/6 \approx 0.0208$
10.
11. 5×10^{-17}
12.
13. $c_5 = f^{(5)}(a)/5!$

□ IS 2

1. 1.49707
2.
3. 6
4.
5. $\frac{1}{e^2-e}$
6.
7. not geometric
8.
9. 1/54
10.
11. 3/11
12.
13. (d) 7.01471 sec.

□ IS 3

1. \$112.75 compounded daily
2.
3. \$61,865.62
4.
5. \$57,413.32
6.
7. \$1,199.10 payment
8.

□ **IS 4**

1. (a) a_n alternates 1 and -1, P_n alternates 1 and 0
   (b) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n$ diverges which is not 0, so series diverges by Divergence Test.
   (c) $\lim_{n\to\infty} P_n$ diverges, so series diverges by definition (of series convergence).
2.
3. (b) $\lim_{n\to\infty} a_n = \frac{1}{2}$, so...
   (c) $\lim_{n\to\infty} P_n = \infty$, so
4.
5. (b) $\lim_{n\to\infty} a_n = 0$, so series convergence is inconclusive.
   (c) $\lim_{n\to\infty} P_n \approx 20.0855$, so series converges by definition.
6.
7. (b) $P_n = \frac{2}{3}\left(1-(-1/2)^{n+1}\right)$, so...
   (c) $E_n = (\frac{-1}{2})^{n+1}$, so...
8.
9. (see idea in 7 solution)

□ **IS 5**

1. $\int_1^\infty \frac{1}{\sqrt{x}}dx = \lim_{b\to\infty} 2b^{(1/2)} - 2 = \infty$
2.
3. Since $\frac{1}{\sqrt{n}} \searrow 0$, series converges by the Alternating Series Thm; $P_{99} \approx -.6550$; $|E_{99}| \leq 0.1$; $L = -0.66 \pm 0.10$
4.
5. series diverges by the Divergence Test
6.
7. $L = -0.15060 \pm 0.00001$
8.

9. integral is 1/4; series converges by Integral Test
10.
11. integral becomes $\lim_{b\to\infty} \frac{b^{(-p+1)}}{-p+1} - \frac{1}{-p+1} = \infty$ since $-p+1>0$

□ IS 6

1. converges by comparison to $\sum \frac{1}{n^4}$
2.
3. diverges by comparison to $\sum \frac{1}{2n}$
4.
5. converges by comparison to $\sum \frac{1}{2^n}$
6.
7. converges by comparison to $\sum \frac{1}{n^{3/2}}$
8.
9. converges by comparison to $\sum \frac{2^n}{e^n}$ or $\sum \frac{n}{n^3}$
10.
11. diverges by comparison to $\sum \frac{5^n}{(2)3^n}$ or Divergence Test
12.
13. diverges by comparison to $\sum \frac{1}{2(n+1)}$
14.
15. converges by comparison to $\sum \frac{1}{n^2}$ or $\sum \frac{1}{e^n}$
16.

□ IS 7

1. $\sum\limits_{n=0}^{\infty} \frac{3^n}{n!} x^n$
2.
3. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}(x-2)^n$; plots appear to converge on about (0,4)
4.
5. (a) $\sum\limits_{n=0}^{\infty} \frac{1}{n!} x^n$

   (b) $\lim\limits_{n\to\infty} |E_n| \leq \lim\limits_{n\to\infty} \frac{e^d}{(n+1)!} |x|^{n+1} = 0$
6.

□ IS 8 Selected Exercises

1.

2.

3.

4.

5.

6.

7. (a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!}$ with interval of convergence $(-\infty, \infty)$

   (b) $\frac{f^{(27)}(0)}{27!}$ =coef of $x^{27} = \frac{(-1)^{13}}{13!}$, so about -1.749×10^{18}

   (c) $f^{(28)}(0) = 0$

8. $1 + t - \frac{1}{3}t^3 - \frac{1}{6}t^4 + \cdots$

9.

10.

□ IS 9

1. 5; (-3,7)

2.

3. 2; (-2,2)

4.

5. 1/3; (2/3,4/3)

6.

7. e; $(-e, e)$

8.

9. (a) 4/3

   (b) $c_n = \frac{3^n}{4^n} c_0$

   (c) $\frac{4c_0}{4-3x}$

10.

11. (a) $f(x) = x - \frac{3}{2}x^3 + \frac{27}{8}x^5 - \frac{135}{16}x^7 + \cdots$

    (b) $1/\sqrt{3}$; $\left(-1/\sqrt{3}, 1/\sqrt{3}\right)$

12.