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Do you think you understand enough about how computers and calculators find numerical solutions for real number problems? Try to compute $\sin(e^{40})$ on your favorite calculator or computer. If you are lucky, it will indicate an error. Even *Mathematica 8* reports wrong answers as witnessed below.

Trig Functions of very large arguments

Enter (Shift + Enter) each input cell to see the resulting computed value:

In[1]:= **Sin[Exp[40]]**

Out[1]= $\sin(e^{40})$

The following answers computed in *Mathematica* 8.0 -- are totally WRONG!

In[2]:= **N[Sin[Exp[40]]]**

Out[2]= -0.130918

In[3]:= **Sin[Exp[40.]]**

Out[3]= -0.130918

- ## Neither of the above are reliable, here's the correct value:

The following asks that the exact value Sin[Exp[40]] be computed with the ANSWER precise to 18 digits.

In[4]:= **N[Sin[Exp[40]], 18]**

Out[4]= 0.948084708486647365

Mathematica operation N[___, m] will give reliable answers by using advanced software algorithms to compute to the desired precision for any m>16. If the m argument is missing (N[___]), or for m<=16, *Mathematica* will use hardware computations where each value is restricted by hardware to have about 16 significant digits. This is not enough digits to compute $\sin(e^{40})$ as we explain below. The incorrect values of the previous section can be traced to this hardware restriction.

- ## You must start with many more digits because of the subtraction of near equals. Let's do it step by step to see the numerical difficulty.

How would you find a sin value for a large argument, such as $181\pi / 6$? Subtract off multiples of 2π. Since $180\pi / 6$ is 30π, this leaves only $1\pi / 6$ and $\sin(181\pi / 6) = \sin(181\pi / 6 - 30\pi) = \sin(\pi / 6) = 1/2$. Let's try the same thing with $\sin(e^{40})$, starting with a value for e^{40} that has 22 significant digits.

In[5]:= **e40 = N[Exp[40], 22]**

Out[5]= $2.353852668370199854079 \times 10^{17}$

First, let's find n so that (n 2 Pi) < Exp[40] < ((n+1)2 Pi).

In[6]:= **e40 / (2 Pi)**

Out[6]= $3.746272874811651419849 \times 10^{16}$

In[7]:= **n = IntegerPart[e40 / (2 Pi)]**

Out[7]= $37\,462\,728\,748\,116\,514$

This is the exact integer n. Let's compute (n 2 Pi) with 22 significant digits.

In[8]:= **n2pi = N[n 2 Pi, 22]**

Out[8]= $2.353852668370199841607 \times 10^{17}$

Now, subtract Exp[40] - (n 2 Pi). Notice that e40 and n2pi agree to 17 digits.

In[9]:= **e40**

Out[9]= $2.353852668370199854079 \times 10^{17}$

In[10]:= **e40 - n2pi**

Out[10]= 1.2472

See the loss of significant digits! But we still get 5 reliable digits in the answer.

In[11]:= **Sin[e40 - n2pi]**

Out[11]= 0.9481

The problem with most calculators and computers, including decimals in Mathematica, is that only about 16 digits are saved in each calculation, so that subtraction of e40 and n2pi leaves only incorrect noise.